Seiberg-Witten-Floer homology of a surface times a circle for non-torsion spin-c structures

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Abstract: We determine the Seiberg–Witten–Floer homology groups of the 3-manifold Σ × S 1, where Σ is a surface of genus g 2, together with its ring structure, for a Spin structure with non-vanishing first Chern class. We give applications to computing Seiberg–Witten invariants of 4-manifolds which are connected sums along surfaces and also we reprove the higher type adjunction inequalities obtained by Oszváth and Szabó. (© 2005 WILEY-VCH Verlag GmbH Co. KGaA, Weinheim)

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SEIBERG-WITTEN-FLOER HOMOLOGY OF A SURFACE TIMES A CIRCLE

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Abstract. We determine the Seiberg-Witten-Floer homology groups of the 3-manifold $\Sigma \times \mathbb{S}^1$, where Σ is a surface of genus $g \geq 1$, together with its ring structure. We give applications to computing Seiberg-Witten invariants of 4-manifolds which are connected sums along surfaces and also we reprove the higher type adjunction inequalities obtained by Oszváth and Szabó [22].

1. Introduction

In this paper we study the gluing theory for Seiberg-Witten invariants of 4-manifolds split along the 3-manifold $Y = \Sigma \times \mathbb{S}^1$, where Σ is a surface of genus $g \geq 1$. This produces applications to the determination of the Seiberg-Witten invariants of 4-manifolds which are constructed as connected sums of other 4-manifolds along embedded surfaces, and to obtain constraints for the Seiberg-Witten invariants of 4-manifolds containing an embedded surface of genus g and non-negative self-intersection. The serminal work in this direction is provided by [13] leading to a proof of the generalized Thom conjecture. Analysis of this kind on non-trivial circle bundles over surfaces appears in [15] [23].

Before stating the results, we set up some notation. Let X be a closed, connected, oriented smooth 4-manifold with $b^+ > 0$ and a fixed homology orientation (i.e. an orientation of $H^1(X;\mathbb{R}) \oplus H^+(X;\mathbb{R})$). For a $\mathrm{Spin}^{\mathbb{C}}$ structure $\mathfrak{s}$, the Seiberg-Witten invariant [28] [24] [14] [26] [27] is a linear functional

$$SW_{X,\mathfrak{s}} : \mathbb{A}(X) \to \mathbb{Z},$$

where $\mathbb{A}(X) = \mathrm{Sym}^* H_0(X) \otimes \Lambda^* H_1(X)$, the free graded algebra generated by the class of the point $x \in H_0(X)$ and the 1-cycles $\gamma \in H_1(X)$ (we understand rational coefficients). We grade $\mathbb{A}(X)$ by declaring the degree of x to be 2 and the degree of the elements in $H_1(X)$

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to be 1. The invariants are constructed by endowing X with a metric g and studying the moduli space $\mathcal{M}_X(\mathfrak{s})$ of solutions (A,Φ) modulo gauge to the Seiberg-Witten equations

$$\left\{ \begin{array}{l} \rho((F_A - \sqrt{-1}\,\xi)^+) = (\Phi\Phi^*)_0 \\ D_A\Phi = 0, \end{array} \right. \tag{1}$$

where Φ is a section of the positive *Spin*^{c}-bundle W^+ of $\mathfrak{s}$, A is a connection on the determinant line bundle $L = \det W^{\pm}$, $D_A : \Gamma(W^+) \to \Gamma(W^-)$ is the Dirac operator twisted with the connection A, ρ denotes Clifford multiplication, $(\Phi\Phi^*)_0$ is the trace-free part of $(\Phi\Phi^*)$ interpreted as an endomorphism of W^+ and ξ is a (small) closed real two-form introduced as a perturbation.

Note that the invariants are zero on elements whose degree is not $d(\mathfrak{s})$ where

$$d(\mathfrak{s}) = \frac{c_1(\mathfrak{s})^2 - (2\chi(X) + 3\sigma(X))}{4}$$

is the dimension of $\mathcal{M}_X(\mathfrak{s})$. When $b^+ > 1$ the Seiberg-Witten invariants are independent of metrics and perturbations. When $b^+ = 1$ the Seiberg-Witten invariants depend on a chamber structure. Fix a component $\mathcal{K}_0$ of the positive cone $\mathcal{K}(X) = \{x \in H^2(X;\mathbb{R}) - \{0\}/x^2 \geq 0\}$. Then we say that the Seiberg-Witten invariants are computed in $\mathcal{K}_0$ when the metric g and perturbation ξ satisfy $(c_1(\mathfrak{s}) + \frac{1}{2\pi}[\xi]) \cdot \omega_g < 0$, where $\omega_g \in H^2(X;\mathbb{Z})$ is a period point for the metric g lying in $\mathcal{K}_0$.

A basic class for X is a Spin$^{\mathbb{C}}$ structure with non-zero Seiberg-Witten invariant. By analogy with the definitions of simple type in the context of Donaldson invariants [19, introduction], we give the following

Definition 1.1. Let X be a 4-manifold with $b^+ > 1$. We say that

- X is of *simple type* if $SW_{X,\mathfrak{s}}(z) = 0$ for any z in the ideal generated by x in $\mathbb{A}(X)$, for any $\mathfrak{s}$.
- X is of *H_1-simple type* if $SW_{X,\mathfrak{s}}(z) = 0$ for any z in the ideal of $\mathbb{A}(X)$ generated by $H_1(X)$, for any $\mathfrak{s}$.
- X is of *strong simple type* if it is both of simple type and of H_1-simple type, i.e. $SW_{X,\mathfrak{s}}(z) = 0$ whenever $\deg(z) > 0$, for any $\mathfrak{s}$.

Note that when X has $b_1 = 0$ it is automatically of H_1-simple type. There are manifolds not of H_1-simple type (for instance any manifold of the connect sum $X \# \mathbb{S}^1 \times \mathbb{S}^3$ along a homologically trivial embedded two sphere, where X has $b^+ > 1$, see [22, proposition 2.2]), but it is an open question whether all 4-manifolds with $b^+ > 1$ are of simple type.

Now we are ready to state our main result. In the one hand, we have applications to computing the Seiberg-Witten invariants of connected sums along surfaces (see [16]). We prove the following results in section 7.

Theorem 1.2. *Let $\bar{X}_1$ and $\bar{X}_2$ be 4-manifolds with embedded surfaces $\Sigma \hookrightarrow \bar{X}_i$, $i = 1, 2$, of the same genus $g \geq 1$, self-intersection zero and representing non-torsion homology classes, and let $X = \bar{X}_1 \#_\Sigma \bar{X}_2$ be their connected sum along Σ. Suppose that $\bar{X}_1$, $\bar{X}_2$ are both of strong simple type, and let $\mathfrak{s}_1$, $\mathfrak{s}_2$ be Spin$^{\mathbb{C}}$ structures on $\bar{X}_1$, $\bar{X}_2$ respectively, such that $c_1(\mathfrak{s}_1) \cdot \Sigma = c_1(\mathfrak{s}_2) \cdot \Sigma$ and $d(\mathfrak{s}_1) = d(\mathfrak{s}_2) = 0$. Let $\mathfrak{s}_o$ be a Spin$^{\mathbb{C}}$ structure on X obtained by gluing $\mathfrak{s}_1$ and $\mathfrak{s}_2$ and let $z \in \mathbb{A}(X)$ such that $d(\mathfrak{s}_o) = \deg z$. Then*

$$\sum_{h \in \mathcal{R}im} SW_{X, \mathfrak{s}_o + h}(z) = \begin{cases} SW_{\bar{X}_1, \mathfrak{s}_1}(1) \cdot SW_{\bar{X}_2, \mathfrak{s}_2}(1) & z = 1,\ |c_1(\mathfrak{s}) \cdot \Sigma| = 2g - 2 \\ 0 & z = 1,\ |c_1(\mathfrak{s}) \cdot \Sigma| < 2g - 2 \\ 0 & \deg(z) > 0 \end{cases}$$

where $\mathcal{R}im \subset H^2(X; \mathbb{Z})$ is the subspace generated by the rim tori. If any of the manifolds has $b^+ = 1$, then the Seiberg-Witten invariants are computed for the component of the positive cone containing $\varepsilon P.D.[\Sigma]$, where $\varepsilon = 1$ if $c_1(\mathfrak{s}_o) \cdot \Sigma < 0$ and $\varepsilon = -1$ if $c_1(\mathfrak{s}_o) \cdot \Sigma \geq 0$.

Moreover if the connected sum is admissible (see definition 7.4), then X is of strong simple type, there are no basic classes $\mathfrak{s}$ of X such that $|c_1(\mathfrak{s}) \cdot \Sigma| < 2g - 2$, and the basic classes of X are in bijection with pairs of basic classes $(\mathfrak{s}_1, \mathfrak{s}_2)$ of X_1 and X_2 respectively, such that $c_1(\mathfrak{s}_1) \cdot \Sigma = c_1(\mathfrak{s}_2) \cdot \Sigma = \pm(2g - 2)$.

This theorem is analogous to [16, corollary 13] and [16, corollary 15] for Donaldson invariants. It is generalised to the following analogue of [19, theorem 7.8],

Theorem 1.3. *Let $\bar{X}_1$ and $\bar{X}_2$ be 4-manifolds with embedded surfaces $\Sigma \hookrightarrow \bar{X}_i$, $i = 1, 2$, of the same genus $g \geq 1$, self-intersection zero and representing non-torsion homology classes, and let $X = \bar{X}_1 \#_\Sigma \bar{X}_2$ be their connected sum along Σ. Suppose that $\bar{X}_1$, $\bar{X}_2$ are of H_1-simple type, and let $\mathfrak{s}_1$, $\mathfrak{s}_2$ be Spin$^{\mathbb{C}}$ structures on $\bar{X}_1$, $\bar{X}_2$ respectively, such that $c_1(\mathfrak{s}_1) \cdot \Sigma = c_1(\mathfrak{s}_2) \cdot \Sigma$. Let $\mathfrak{s}_o$ be a Spin$^{\mathbb{C}}$ structure on X obtained by gluing $\mathfrak{s}_1$ and $\mathfrak{s}_2$ and let $z \in \mathbb{A}(X)$ such that $d(\mathfrak{s}_o) = \deg z$. Then*

$$\sum_{h \in \mathcal{R}im} SW_{X, \mathfrak{s}_o + h}(z) = 0$$

if $\deg(z) > 0$ or if $c_1(\mathfrak{s}) \cdot \Sigma \not\equiv 2g - 2 \pmod 4$. If X has $b^+ = 1$ then the Seiberg-Witten invariants are computed for the component of the positive cone containing $\varepsilon P.D.[\Sigma]$, where $\varepsilon = 1$ if $c_1(\mathfrak{s}_o) \cdot \Sigma < 0$ and $\varepsilon = -1$ if $c_1(\mathfrak{s}_o) \cdot \Sigma \geq 0$.

Moreover if the connected sum is admissible then X is of strong simple type and there are no basic classes $\mathfrak{s}$ of X unless $c_1(\mathfrak{s}) \cdot \Sigma \equiv 2g - 2 \pmod 4$.

In the other hand, our analysis also leads to a different proof of the higher type adjunction inequalities first obtained by Oszváth and Szabó in [22]. Our method of proof is more transparent and parallels that of [20] for proving the higher type adjunction inequalities for Donaldson invariants. Section 8 is devoted to this issue.

Theorem 1.4. *([22, theorem 1.4]) Let X be a 4-manifold and let $\Sigma \subset X$ be an embedded surface of genus $g \geq 1$ representing a non-torsion homology class with self-intersection $\Sigma^2 \geq 0$. Let $a \in \mathbb{A}(X)$ and $b \in \mathbb{A}(\Sigma)$. If X has $b^+ > 1$ and $\mathfrak{s}$ is a Spin$^{\mathbb{C}}$ structure with $SW_{X,\mathfrak{s}}(a\,b) \neq 0$ then we have*

$$|c_1(\mathfrak{s}) \cdot \Sigma| + \Sigma^2 + \deg(b) \leq 2g - 2.$$

Furthermore, when $b^+ = 1$ then for each Spin$^{\mathbb{C}}$ structure $\mathfrak{s}$ on X with $-c_1(\mathfrak{s}) \cdot \Sigma + \Sigma^2 \geq 0$, for which $SW_{X,\mathfrak{s}}(ab) \neq 0$, when calculated in the component of $\mathcal{K}(X)$ containing P.D.$[\Sigma]$, we have an inequality

$$-c_1(\mathfrak{s}) \cdot \Sigma + \Sigma^2 + \deg(b) \leq 2g - 2.$$

Theorem 1.5. *([22, theorem 1.1]) Let X be a 4-manifold of H_1-simple type (e.g. with $b_1 = 0$) and let $\Sigma \subset X$ be an embedded surface of genus $g \geq 1$ representing a non-torsion homology class with $\Sigma^2 \geq 0$. If X has $b^+ > 1$ then for each basic class $\mathfrak{s}$ for X we have*

$$|c_1(\mathfrak{s}) \cdot \Sigma| + \Sigma^2 + 2d(\mathfrak{s}) \leq 2g - 2.$$

If $b^+ = 1$ then for each basic class $\mathfrak{s}$ for the Seiberg-Witten invariants of X calculated in the component of $\mathcal{K}(X)$ which contains P.D.$[\Sigma]$ such that $-c_1(\mathfrak{s}) \cdot \Sigma + \Sigma^2 \geq 0$, we have an inequality

$$-c_1(\mathfrak{s}) \cdot \Sigma + \Sigma^2 + 2d(\mathfrak{s}) \leq 2g - 2.$$

□

This is a particular case of the more general result

Theorem 1.6. *([22, theorem 1.3]) Let X be a 4-manifold with an embedded surface $\imath : \Sigma \hookrightarrow X$ of genus $g \geq 1$ representing a non-torsion homology class with $\Sigma^2 \geq 0$. Let l be an integer so that there is a symplectic basis $\{\gamma_i\}_{i=1}^{2g}$ of $H_1(\Sigma)$ with $\gamma_i \cdot \gamma_{g+i} = 1$, $1 \leq i \leq g$, satisfying that $\imath_*(\gamma_j) = 0$ in $H_1(X)$ for $i = 1, \ldots, l$. Let $a \in \mathbb{A}(X)$ and $b \in \mathbb{A}(\Sigma)$ be an element of degree $\deg(b) \leq l + 1$. If X has $b^+ > 1$ and $\mathfrak{s}$ is a Spin$^{\mathbb{C}}$ structure such that $SW_{X,\mathfrak{s}}(a\,b) \neq 0$ then we have*

$$|c_1(\mathfrak{s}) \cdot \Sigma| + \Sigma^2 + 2\deg(b) \leq 2g - 2.$$

Furthermore, when $b^+ = 1$ then for each Spin$^{\mathbb{C}}$ structure $\mathfrak{s}$ on X with $-c_1(\mathfrak{s}) \cdot \Sigma + \Sigma^2 \geq 0$, for which $SW_{X,\mathfrak{s}}(a\,b) \neq 0$, when calculated in the component of $\mathcal{K}(X)$ containing P.D.$[\Sigma]$, we have

$$-c_1(\mathfrak{s}) \cdot \Sigma + \Sigma^2 + 2\deg(b) \leq 2g - 2.$$

In order to prove these results, we need to determine completely the structure of the Seiberg-Witten-Floer (co)homology of the three manifold $Y = \Sigma \times \mathbb{S}^1$, where Σ is a closed surface of genus $g \geq 1$. By using the intrinsic properties of this homology gathered in [4] and the Seiberg-Witten invariants of the ruled surface $S = \Sigma \times \mathbb{S}^2$ we prove

Theorem 1.7. *Let $\mathfrak{s}_Y$ be a Spin$^{\mathbb{C}}$ structure on $Y = \Sigma \times \mathbb{S}^1$. If $c_1(\mathfrak{s}_Y) \neq 2rP.D.[\mathbb{S}^1]$, with $-(g-1) \leq r \leq g-1$ then $HFSW^*(Y, \mathfrak{s}_Y) = 0$. Let $\mathfrak{s}_r$ be the Spin$^{\mathbb{C}}$ structure on Y with $c_1(\mathfrak{s}_r) = 2rP.D.[\mathbb{S}^1]$, $-(g-1) \leq r \leq g-1$, and set $d = g-1-|r|$. For $r \neq 0$, $HFSW^*(Y, \mathfrak{s}_r) \cong H^*(s^d\Sigma)$ as vector spaces (where $s^d\Sigma$ is the d-th symmetric product of Σ). For $r = 0$ the Floer-Seiberg-Witten homology of the trivial Spin$^{\mathbb{C}}$ structure on Y with perturbation $t[\omega] \neq 0$, where $[\omega] = PD[\mathbb{S}^1]$ is $HFSW^*_{t[\omega]}(Y, \mathfrak{s}_0) \cong H^*(s^{g-1}\Sigma) \otimes \mathbb{Q}((q))$.* $\square$

Theorem 1.7 follows from theorem 4.2, corollary 4.5 and corollary 6.3. The Seiberg-Witten-Floer homology of $Y = \Sigma \times \mathbb{S}^1$ has a natural ring structure coming from the cobordism which is a pair of pants times Σ (cf. [6]). This should be closely related (if not isomorphic) to the quantum cohomology of the symmetric products of Σ (see [1] for a partial computation of the latter), in the same way as the instanton Floer cohomology of $\Sigma \times \mathbb{S}^1$ is isomorphic to the quantum cohomology of the moduli space of rank 2 odd degree stable vector bundles over Σ (see [18]). We fully compute the ring structure of $HFSW^*(Y, \mathfrak{s}_r)$, which is the deformation of the ring structure on $H^*(s^{g-1-|r|}\Sigma)$.

Theorem 1.8. *Let $-(g-1) \leq r \leq g-1$ and set $d = g-1-|r|$. If $r \neq 0$, let $\hat{V}_r = HFSW^*(Y, \mathfrak{s}_r) \otimes \mathbb{Q}((q))$, where the new variable q has degree $-|r|$ and has been introduced such that $\hat{V}_r$ becomes a $\mathbb{Z}$-graded ring. If $r = 0$, let $\hat{V}_0 = HFSW^*_{t[\omega]}(Y, \mathfrak{s}_0)$, as a vector space over $\mathbb{Q}((q))$, naturally endowed with an integer grading where q has degree 0. In both cases, there is a presentation*

$$\hat{V}_r = \bigoplus_{k=0}^{d} \Lambda_0^k \otimes \frac{\mathbb{Q}((q))[\eta, \theta]}{(\tilde{\mathcal{R}}_k^g, \theta \tilde{\mathcal{R}}_{k+1}^g)},$$

where

$$\tilde{\mathcal{R}}_k^g = \sum_{i=0}^{\alpha} \frac{\binom{d-k-\alpha+1}{i}}{i!\binom{g-k}{i}} (-1)^i \eta^{\alpha-i}\theta^i - \sum_{i=0}^{\alpha+|r|} \frac{\binom{\alpha+|r|}{i}}{i!\binom{g-k}{i}} \eta^{\alpha+|r|-i}\theta^i\, q^2,$$

where $\alpha = [\frac{d-k}{2}] + 1$, $0 \leq k \leq d$, and $\tilde{\mathcal{R}}_{d+1}^g = 1$. $\square$

This follows from remark 5.9 and theorem 6.7.

2. Review of Seiberg-Witten-Floer homology: the case $c_1(\mathfrak{s}_Y)$ non-torsion

Let Y be an oriented 3-manifold with first Betti number $b_1 > 0$ and a Spin$^{\mathbb{C}}$ structure $\mathfrak{s}_Y$ on Y with $c_1(\mathfrak{s}_Y)$ non-torsion. We are going to review the construction of the Seiberg-Witten-Floer (co)homology groups $HFSW_*(Y, \mathfrak{s}_Y)$ of Y from [4].

2.1. Definition of $HFSW(Y, \mathfrak{s}_Y)$. We endow Y with a metric g_Y and fix a base connection A_0 on the determinant line bundle $L_Y = \det W_Y$, where W_Y stands for the spin bundle. There is a Chern-Simons Seiberg-Witten functional (taking values in $\mathbb{R}/\mathbb{Z}$)

$$\mathcal{C}_\eta(A, \psi) = -\frac{1}{2}\left(\int_Y (A - A_0) \wedge (F_A + F_{A_0} - 2 *_Y \sqrt{-1}\eta) + \langle \psi, D_A\psi \rangle dvol_Y\right),$$

for a connection A on L_Y and a section ψ of W_Y, where D_A stands for the Dirac operator on W_Y coupled with A, and η is a (perturbative) real coexact one-form on Y. The critical points of $\mathcal{C}_\eta$ correspond to translation invariant solutions of (1) on the tube $Z = Y \times \mathbb{R}$ for the perturbation $\xi = *\eta$. The equations we are led to are

$$\left\{ \begin{array}{l} *F_A = q(\psi) + \eta \\ D_A \psi = 0, \end{array} \right. \tag{2}$$

where $q(\psi)$ is the standard quadratic form. When $c_1(\mathfrak{s}_Y)$ is non-torsion we can choose a generic coexact perturbation η to get a finite collection of non-degenerate irreducible (oriented) points [4]. There is also defined an index $\mathrm{ind}(a)$ for every solution $a = (A, \psi)$ of (2), taking values in $\mathbb{Z}/N\mathbb{Z}$, where $\langle c_1(\mathfrak{s}_Y), H^1(Y;\mathbb{Z})\rangle = N\mathbb{Z}$. The Seiberg-Witten-Floer chain complex $CFSW_*(Y)$ will be the vector space generated by the solutions to (2) with this grading.

Let a and b be two solutions of (2). For generic perturbations [4], the moduli space $\hat{\mathcal{M}}(a,b)$ of Seiberg-Witten solutions on the tube $Y \times \mathbb{R}$ with limits a and b respectively, is smooth, orientable and admits a free $\mathbb{R}$-action with quotient $\mathcal{M}(a,b)$. We shall denote by $\mathcal{M}^D(a,b)$ the component of dimension D. Note that $D \equiv \mathrm{ind}(b) - \mathrm{ind}(a) - 1 \pmod N$. We define a boundary map

$$\begin{array}{rcl} \partial : CFSW_i(Y) & \to & CFSW_{i-1}(Y) \\ a & \mapsto & \displaystyle\sum_{\substack{b \\ \mathrm{ind}(b)=\mathrm{ind}(a)-1}} \#\mathcal{M}^0(a,b)\, b. \end{array}$$

It is a fact that $\partial^2 = 0$ [4] so we are allowed to give the following

Definition 2.1. Suppose $c_1(\mathfrak{s}_Y)$ is not torsion. The *Seiberg-Witten-Floer homology* of Y, $HFSW_*(Y, \mathfrak{s}_Y)$, is the homology of the complex $(CFSW_*(Y), \partial)$. The Seiberg-Witten-Floer cohomology, $HFSW^*(Y, \mathfrak{s}_Y)$, is the homology of the dual complex.

This homology is independent of metrics and of (small) perturbations [4]. There is a natural isomorphism $HFSW^*(Y, \mathfrak{s}_Y) \cong HFSW_{-*}(-Y, -\mathfrak{s}_Y)$, where $-Y$ is Y with reversed orientation, which yields an intersection pairing

$$\langle , \rangle : HFSW^*(Y, \mathfrak{s}_Y) \otimes HFSW^{-*}(-Y, -\mathfrak{s}_Y) \to \mathbb{Q}.$$

2.2. Action of $\mathbb{A}(Y)$ on $HFSW_*(Y, \mathfrak{s}_Y)$. There is a natural action of the homology (in degrees 0 and 1) on the Seiberg-Witten-Floer homology (usually called the μ map). Let $\alpha \in H_{2-j}(Y)$ be the point or a 1-cycle. We have cycles V_α, in the moduli spaces $\hat{\mathcal{M}}(a,b)$, of codimension j, representing $\mu(\alpha \times t)$, for $\alpha \times t \subset Y \times \mathbb{R}$. Using them, we construct a map

$$\begin{array}{rcl} \mu(\alpha) : CFSW_i(Y) & \to & CFSW_{i-j}(Y) \\ a & \mapsto & \displaystyle\sum_{\substack{b \\ \mathrm{ind}(b)=\mathrm{ind}(a)-j}} \#(\hat{\mathcal{M}}^j(a,b) \cap V_\alpha)\, b, \end{array}$$

where $\hat{\mathcal{M}}^j(a,b)$ is the j-dimensional component of $\hat{\mathcal{M}}(a,b)$. To see that $\partial\circ\mu(\alpha)+\mu(\alpha)\circ\partial = 0$, consider the 1-dimensional moduli space $\hat{\mathcal{M}}^{j+1}(a,c)\cap V_\alpha$, for $\mathrm{ind}(c) = \mathrm{ind}(a)-j-1$. Then the number of points in the boundary of its compactification, counted with signs, is

$$\sum_{\substack{b\\ \mathrm{ind}(b)=\mathrm{ind}(a)-j}} \#(\hat{\mathcal{M}}^j(a,b)\cap V_\alpha)\cdot \#\mathcal{M}^0(b,c) + \sum_{\substack{b\\ \mathrm{ind}(b)=\mathrm{ind}(a)-1}} \#\mathcal{M}^0(a,b)\cdot\#(\hat{\mathcal{M}}^j(b,c)\cap V_\alpha) = 0.$$

So $\mu(\alpha)$ descends to a well-defined map

$$\mu(\alpha): HFSW_*(Y,\mathfrak{s}_Y)\to HFSW_{*-j}(Y,\mathfrak{s}_Y).$$

This provides an action of $\mathbb{A}(Y) = \mathrm{Sym}^* H_0(Y)\otimes\Lambda^* H_1(Y)$ on $HFSW_*(Y,\mathfrak{s}_Y)$.

2.3. Relative Seiberg-Witten invariants and gluing formula. Let X_1 be an oriented, connected four-manifold furnished with a cylindrical end of the form $Y\times[0,\infty)$. Suppose we have a $\mathrm{Spin}^{\mathbb{C}}$ structure $\mathfrak{s}_1$ over X_1 whose restriction to Y is $\mathfrak{s}_Y$. Consider the moduli spaces $\mathcal{M}_{X_1}(a)$ of solutions to the (perturbed) Seiberg-Witten equations on X_1 with asymptotic limit $a\in CFSW_*(Y)$. Genericaly, $\mathcal{M}_{X_1}(a)$ has infinitely many components, we denote by $\mathcal{M}^k_{X_1}(a)$ the components of dimension k, then $\mathcal{M}^k_{X_1}(a)$ has a natural compactification by adding lower dimensional strata. For any $z_1\in\mathbb{A}(X_1)$, we have a codimenion $deg(z_1)$ subset V_{z_1} in $\mathcal{M}^k_{X_1}(a)$. In particular, $\mathcal{M}^{deg(z_1)}_{X_1}(a)\cap V_{z_1}$ consists of finitely many oriented points, hence we have an element

$$\phi^{SW}_{X_1}(\mathfrak{s}_1,z_1) = \sum_a \#(\mathcal{M}^{deg(z_1)}_{X_1}(a)\cap V_{z_1})\ a\in CFSW_*(Y,\mathfrak{s}_Y),$$

which is actually a cycle, and defines a Seiberg-Witten-Floer homology class $\phi^{SW}_{X_1}(\mathfrak{s}_1,z_1)\in HFSW_*(Y,\mathfrak{s}_Y)$ independent of metrics and perturbations. Also for $\alpha\in H_0(Y)$ or $\alpha\in H_1(Y)$, we have $\mu(\alpha)\phi^{SW}_{X_1}(\mathfrak{s}_1,z) = \phi^{SW}_{X_1}(\mathfrak{s}_1,\alpha z)$.

For a second four-manifold X_2 with a cylindrical end $(-Y)\times[0,\infty)$, we construct $X = X_1\cup_Y X_2$ by cutting the ends and gluing along the common boundary Y. The resulting manifold may depend on the isotopy class of the diffeomorphism identifying the boundaries, but we shall not make the dependence explicit. If there is a $\mathrm{Spin}^{\mathbb{C}}$ structure $\mathfrak{s}_2$ on X_2 with $\mathfrak{s}_2|_Y\cong\mathfrak{s}_Y$, then we can glue $\mathfrak{s}_1$ and $\mathfrak{s}_2$. The indeterminacy for the gluing is parametrised by $\mathrm{coker}(H^1(X_1;\mathbb{Z})\oplus H^1(X_2;\mathbb{Z})\to H^1(Y;\mathbb{Z}))$.

Theorem 2.2 ([4])**.** *Let X be a closed manifold with $b^+\geq 1$ which is written as $X = X_1\cup_Y X_2$, where X_1 and X_2 are 4-manifolds with boundary and $\partial X_1 = -\partial X_2 = Y$. Suppose that we have $\mathrm{Spin}^{\mathbb{C}}$ structures $\mathfrak{s}_1$ and $\mathfrak{s}_2$ on X_1 and X_2 respectively such that $\mathfrak{s}_1|_Y\cong\mathfrak{s}_2|_Y\cong\mathfrak{s}_Y$. Then we have the following gluing formula for $z_i\in\mathbb{A}(X_i)$, $i=1,2$,*

$$\sum_{\{\mathfrak{s}/\mathfrak{s}|_{X_i}=\mathfrak{s}_i,\ i=1,2\}} SW_{X,\mathfrak{s}}(z_1z_2) = \langle\phi^{SW}_{X_1}(\mathfrak{s}_1,z_1),\phi^{SW}_{X_2}(\mathfrak{s}_2,z_2)\rangle.$$

When $b^+=1$, the Seiberg-Witten invariants correspond to a metric giving a long tube.

3. Review of Seiberg-Witten-Floer homology: the case $c_1(\mathfrak{s}_Y)$ torsion

Let Y be an oriented 3-manifold with first Betti number $b_1 > 0$ and a $\mathrm{Spin}^{\mathbb{C}}$ structure $\mathfrak{s}_Y$ on Y with $c_1(\mathfrak{s}_Y)$ torsion. Let us review the construction of the Seiberg-Witten-Floer (co)homology groups $HFSW_*(Y, \mathfrak{s}_Y)$ of Y from [11]. Note that a coexact perturbation η for the Chern-Simons Seiberg-Witten functional as in section 2 would introduce the reducible monopoles to the critical points of the $\mathcal{C}_\eta$, which would cause serious analytical problems. So we have to introduce the perturbation with non-trivial cohomology class.

3.1. Definition of $HFSW(Y, \mathfrak{s}_Y)$. We endow Y with a metric g_Y and fix a base connection A_0 on the determinant line bundle $L_Y = \det W_Y$, where W_Y stands for the spin bundle. There is a Chern-Simons Seiberg-Witten functional (taking values in $\mathbb{R}/\mathbb{Z}$)

$$\mathcal{C}_\eta(A, \psi) = -\frac{1}{2}\left(\int_Y (A - A_0) \wedge (F_A + F_{A_0} - 2 *_Y \sqrt{-1}\eta) + \langle \psi, D_A\psi \rangle dvol_Y\right),$$

for a connection A on L_Y and a section ψ of W_Y, where η is a real-valued one-form on Y with $[*\eta] \neq 0$ in $H^2(Y, \mathbb{R})$. We assume that $[*\eta] = t[\omega]$ where $t < 0$ and $[\omega]$ is Poincare dual to some 1-cycle C in Y. For generic η, $\mathcal{C}_\eta$ has a finite collection of non-degenerate irreducible (oriented) points [4]. There is also defined an index $\mathrm{ind}(a)$ for every critical point $a = (A, \psi)$, taking values in $\mathbb{Z}$. The Seiberg-Witten-Floer chain complex $CFSW_*(Y)$ will be the vector space generated by the critical points over the coefficient field of formal Laurent series $\mathbb{Q}((q)) = \mathbb{Q}[[q]][q^{-1}]$, with this grading.

Let a and b be two critical points of $\mathcal{C}_\eta$. For generic perturbations [4], the moduli space $\hat{\mathcal{M}}(a, b)$ of Seiberg-Witten solutions on the tube $Y \times \mathbb{R}$ with limits a and b respectively, is smooth, orientable and admits a free $\mathbb{R}$-action with quotient $\mathcal{M}(a, b)$. Note that $\mathcal{M}(a, b)$ has infinitely many components with dimension given by $\mathrm{ind}(b) - \mathrm{ind}(a)$ due to the fact that $\mathcal{C}_\eta$ takes values in $\mathbb{R}/\mathbb{Z}$. We define the energy map $\hat{\mathcal{M}}(a, b) \to \mathbb{Z}$ by assigning $\langle [\frac{\sqrt{-1}}{2\pi} F_A], [C \times \mathbb{R}] \rangle$ to any solution $[A, \Phi] \in \hat{\mathcal{M}}(a, b)$, this energy essentially measures the variation of $\mathcal{C}_\eta$ on $Y \times \mathbb{R}$ up to a positive scale. Denote by $\hat{\mathcal{M}}^{(n)}(a, b)$ the preimage of $n \in \mathbb{Z}$ by the energy map, and by $\mathcal{M}^{(n)}(a, b)$ its quotient by $\mathbb{R}$. From the definition of energy, we see that $\mathcal{M}^{(n)}(a, b)$ is empty for $n \ll 0$. $\mathcal{M}^{(n)}(a, b)$ has the standard compactification according to the energy distribution. In particular, the following boundary operator is well-defined:

$$\begin{aligned} \partial : CFSW_i(Y) &\to CFSW_{i-1}(Y) \\ a &\mapsto \sum_{\substack{n \in \mathbb{Z} \\ b, \mathrm{ind}(b) = \mathrm{ind}(a) - 1}} \#\mathcal{M}^{(n)}(a, b) q^n\, b. \end{aligned}$$

The fact that $\partial^2 = 0$ allows us to give the following

Definition 3.1. Suppose that $c_1(\mathfrak{s}_Y)$ is torsion. The *Seiberg-Witten-Floer homology* of Y, $HFSW_*(Y, \mathfrak{s}_Y)$, is the homology of the complex $(CFSW_*(Y), \partial)$. The Seiberg-Witten-Floer cohomology, $HFSW^*(Y, \mathfrak{s}_Y)$, is the homology of the dual complex over $\mathbb{Q}((q))$.

This homology is independent of metrics and of (small) perturbations. There is a natural isomorphism $HFSW^*(Y, \mathfrak{s}_Y) \cong HFSW_{-*}(-Y, -\mathfrak{s}_Y)$, where $-Y$ is Y with reversed orientation, which yields an intersection pairing

$$\langle , \rangle : HFSW^*(Y, \mathfrak{s}_Y) \otimes HFSW^{-*}(-Y, -\mathfrak{s}_Y) \to \mathbb{Q}((q)).$$

3.2. Action of $\mathbb{A}(Y)$ on $HFSW_*(Y, \mathfrak{s}_Y)$. Let $\alpha \in H_{2-j}(Y)$ be the point or a 1-cycle. We have cycles V_α, in the moduli spaces $\hat{\mathcal{M}}^{(n)}(a,b)$, of codimension j, representing $\mu(\alpha \times t)$, for $\alpha \times t \subset Y \times \mathbb{R}$. Then

$$\begin{array}{rcl} \mu(\alpha) : CFSW_i(Y) & \to & CFSW_{i-j}(Y) \\ a & \mapsto & \displaystyle\sum_{\substack{n \in \mathbb{Z} \\ b, \mathrm{ind}(b) = \mathrm{ind}(a) - j}} \#(\hat{\mathcal{M}}^{(n)}(a,b) \cap V_\alpha) q^n \, b \end{array}$$

descends to a well-defined map $\mu(\alpha) : HFSW_*(Y, \mathfrak{s}_Y) \to HFSW_{*-j}(Y, \mathfrak{s}_Y)$. This provides an action of $\mathbb{A}(Y)$ on $HFSW_*(Y, \mathfrak{s}_Y)$.

3.3. Relative Seiberg-Witten invariants and gluing formula. Let X_1 be an oriented, connected four-manifold furnished with a cylindrical end of the form $Y \times [0, \infty)$. Suppose that we have a $\mathrm{Spin}^{\mathbb{C}}$ structure $\mathfrak{s}_1$ over X_1 whose restriction to Y is a torsion $\mathrm{Spin}^{\mathbb{C}}$ structure $\mathfrak{s}_Y$. Considering the moduli spaces $\mathcal{M}_{X_1}(a)$ of solutions to the (perturbed) Seiberg-Witten equations on X_1 with asymptotic limit $a \in CFSW_*(Y)$, the perturbation has to be consistent with the perturbation in the definition of $HFSW_*(Y, \mathfrak{s}_Y)$, so we require that $C = PD([\omega]) \in H_1(Y)$ bounds a surface D in X_1. Generically, $\mathcal{M}_{X_1}(a)$ (if non-empty) is a smooth, oriented manifold of fixed dimension with infinitely many components. We choose a 2-cycle $D \subset X_1$ with $\partial D = C$. Define the energy map as the relative Chern class of $[A, \Phi]$ restricted to (D, C),

$$\mathcal{E} : \mathcal{M}_{X_1}(a) \to H^2(D, C; \mathbb{Z}) \cong \mathbb{Z},$$

which can be realised by assigning $\langle [\frac{\sqrt{-1}}{2\pi} F_A], [D] \rangle$ to each $[A, \Phi] \in \mathcal{M}_{X_1}(a)$. Note that $\mathcal{E}$ is bounded from below on $\mathcal{M}_{X_1}(a)$ (using Weitzenböck formula and the Seiberg-Witten equations). Denote by $\mathcal{M}^{(n)}_{X_1}(a)$ the preimage of n under $\mathcal{E}$. For $n \ll 0$, $\mathcal{M}^{(n)}_{X_1}(a)$ is empty, so we have a well-defined element

$$\phi^{SW}_{X_1}(\mathfrak{s}_1, z_1) = \sum_{\substack{a \\ n \in \mathbb{Z}}} \#(\mathcal{M}^{(n)}_{X_1}(a) \cap V_{z_1}) q^n \, a \in CFSW_*(Y, \mathfrak{s}_Y),$$

which defines a Seiberg-Witten-Floer homology class $\phi^{SW}_{X_1}(\mathfrak{s}_1, z_1) \in HFSW_*(Y, \mathfrak{s}_Y)$ independent of metrics and perturbations. The dependence on the choice of D is only up to a constant power of q. These relative invariants also satisfy $\mu(\alpha)\phi^{SW}_{X_1}(\mathfrak{s}_1, z) = \phi^{SW}_{X_1}(\mathfrak{s}_1, \alpha z)$.

Theorem 3.2 ([11]). *Let X be a closed manifold with $b^+ \geq 1$ which is written as $X = X_1 \cup_Y X_2$, where X_1 and X_2 are 4-manifolds with boundary and $\partial X_1 = -\partial X_2 = Y$. Suppose that we have $\mathrm{Spin}^{\mathbb{C}}$ structures $\mathfrak{s}_1$ and $\mathfrak{s}_2$ on X_1 and X_2 respectively such that*

$\mathfrak{s}_1|_Y \cong \mathfrak{s}_2|_Y \cong \mathfrak{s}_Y$ is a torsion Spin$^{\mathbb{C}}$ structure on Y. Fix a 1-cycle $C \subset Y$ such that there are two 2-cycles $D_i \subset X_i$ with $\partial D_1 = -\partial D_2 = C$, and put $D = D_1 + D_2 \in H_2(X)$. Then we have the following gluing formula for $z_i \in \mathbb{A}(X_i)$, $i = 1, 2$,

$$\sum_{\{\mathfrak{s}/\mathfrak{s}|_{X_i}=\mathfrak{s}_i,\ i=1,2\}} SW_{X,\mathfrak{s}}(z_1 z_2) q^{c_1(\mathfrak{s})\cdot D} = \langle \phi^{SW}_{X_1}(\mathfrak{s}_1, z_1), \phi^{SW}_{X_2}(\mathfrak{s}_2, z_2)\rangle.$$

When $b^+ = 1$, the Seiberg-Witten invariants correspond to a metric giving a long tube.

4. Seiberg-Witten-Floer homology of $\Sigma \times \mathbb{S}^1$

From now on we shall consider the three-manifold $Y = \Sigma \times \mathbb{S}^1$, which is the central object of our study. As $H^2(Y;\mathbb{Z})$ has no 2-torsion, the Spin$^{\mathbb{C}}$ structures $\mathfrak{s}_Y$ on Y are determined by the determinant line bundle $L_Y = c_1(\mathfrak{s}_Y)$. As $c_1(\mathfrak{s}_Y)$ reduces to $w_2(Y) = 0$ modulo 2, it has to be an even class in $H^2(Y;\mathbb{Z})$.

Proposition 4.1. *Let $\mathfrak{s}_Y$ be a Spin$^{\mathbb{C}}$ structure on Y. Let $\mathcal{M}$ be the moduli space of solutions to (2) with perturbation $\eta = 0$. Then $\mathcal{M}$ is empty unless $c_1(\mathfrak{s}_Y) = 2r P.D.[\mathbb{S}^1]$, with $-(g-1) \le r \le g-1$. For $c_1(\mathfrak{s}_Y) = 2rP.D.[\mathbb{S}^1]$, with $-(g-1) \le r \le g-1$ and $r \neq 0$, $\mathcal{M}$ is Morse-Bott irreducible and isomorphic to $s^d\Sigma$ with $d = g - 1 - |r|$.*

Proof. We choose a rotation invariant metric for Y of the form $g_\Sigma + d\theta \otimes d\theta$, where g_Σ is a metric on Σ with unit area and scalar curvature $-4\pi(2g-2)$, and θ is the coordinate on $\mathbb{S}^1 = \mathbb{R}/\mathbb{Z}$. Think of $\Sigma \times \mathbb{S}^1$ as $\Sigma \times [0,1]$ with the boundaries $\Sigma \times \{0\}$ and $\Sigma \times \{1\}$ identified by the identity. The line bundle L_Y is constructed out of the pull back under the projection $\Sigma \times [0,1] \to \Sigma$ of a line bundle L_Σ on Σ by gluing along the boundaries with an isomorphism $\sigma \in \mathcal{G}_\Sigma = \mathrm{Map}(\Sigma, \mathbb{S}^1)$. Then $c_1(L_Y) = c_1(L_\Sigma) + [\sigma] \otimes [\mathbb{S}^1]$, where $[\sigma]$ is the class of σ in $[\Sigma; \mathbb{S}^1] \cong H^1(\Sigma;\mathbb{Z})$. The Spin$^{\mathbb{C}}$ structure $\mathfrak{s}_Y$ induces a Spin$^{\mathbb{C}}$ structure $\mathfrak{s}_\Sigma$ on Σ with determinant line bundle L_Σ. The spin bundle of $\mathfrak{s}_\Sigma$ is of the form $W_\Sigma = (\Lambda^0 \oplus \Lambda^{0,1}) \otimes \mu$, for a line bundle μ such that $L_\Sigma = K_\Sigma^{-1} \otimes \mu^2$, where K_Σ stands for the canonical bundle of Σ.

Now consider any solution (A, ψ) to (2) with $\eta = 0$. In $\Sigma \times [0,1]$, we can kill, by a gauge transformation, the $d\theta$ component of A, i.e. we can suppose that we have a family A_θ, $\theta \in [0,1]$, of connections on L_Σ (up to a constant gauge) with the boundary condition $A_1 = \sigma^*(A_0)$, for some $\sigma \in \mathcal{G}_\Sigma$ in the homotopy class determined by L_Y. So (A,ψ) is interpreted as a path $(A_\theta, \alpha_\theta, \beta_\theta)$, $\theta \in [0,1]$, where $\alpha_\theta \in \Lambda^0 \otimes \mu$ and $\beta_\theta \in \Lambda^{0,1} \otimes \mu$. Let us rewrite equations (2) in this set-up. Clearly $*F_A = \Lambda F_{A_\theta} d\theta + *_\Sigma(\frac{\partial A}{\partial \theta})$, and the map q and the Dirac operator are as follows

$$q(\psi) = -*\frac{\left(\alpha\overline{\beta} + \overline{\alpha}\beta\right)}{2} + \frac{|\alpha|^2 - |\beta|^2}{2} d\theta,$$

$$D_A = \begin{pmatrix} -\sqrt{-1}\frac{\partial}{\partial\theta} & \sqrt{2}\bar{\partial}_A^* \\ \sqrt{2}\bar{\partial}_A & -\sqrt{-1}\frac{\partial}{\partial\theta} \end{pmatrix} : (\Lambda^0 \otimes \mu) \oplus (\Lambda^{0,1} \otimes \mu) \to (\Lambda^0 \otimes \mu) \oplus (\Lambda^{0,1} \otimes \mu).$$

So the solutions to (2) correspond to solutions to

$$\left\{ \begin{array}{l} \frac{\partial\alpha}{\partial\theta} = -\sqrt{-1}\sqrt{2}\bar{\partial}^*_{A_\theta}\beta \\ \frac{\partial\beta}{\partial\theta} = \sqrt{-1}\sqrt{2}\bar{\partial}_{A_\theta}\alpha \\ 2\frac{\partial A_\theta}{\partial\theta} = -\sqrt{-1}(\alpha\overline{\beta} + \beta\overline{\alpha}) \\ 2\sqrt{-1}\Lambda F_{A_\theta} = -|\alpha|^2 + |\beta|^2 \end{array} \right. \tag{3}$$

We can write $A_\theta = \partial_{A_\theta} + \bar{\partial}_{A_\theta}$, so the third line iss $\frac{\partial}{\partial\theta}(\partial_{A_\theta}) = -\frac{\sqrt{-1}}{2}\alpha\overline{\beta}$, $\frac{\partial}{\partial\theta}(\bar{\partial}_{A_\theta}) = -\frac{\sqrt{-1}}{2}\overline{\alpha}\beta$. Now suppose we have a solution to (3). Then we work out the following expression (using $\sqrt{-1}\bar{\partial}^* = \Lambda\partial$ on $(0,1)$-forms and $|\beta|^2 = -\sqrt{-1}\Lambda\beta\wedge\overline{\beta}$)

$$\frac{\partial}{\partial\theta}(\bar{\partial}^*\beta) = -\frac{\partial}{\partial\theta}(\sqrt{-1}\Lambda\partial)\beta + \bar{\partial}^*(\frac{\partial\beta}{\partial\theta}),$$

with the given equalities to get

$$-\frac{1}{\sqrt{2}\sqrt{-1}}\frac{\partial^2\alpha}{\partial\theta^2} = \sqrt{-1}\Lambda\frac{\sqrt{-1}}{2}\alpha\overline{\beta}\beta - \bar{\partial}^*(-\sqrt{-1}\sqrt{2}\bar{\partial}\alpha),$$

$$-\frac{\partial^2\alpha}{\partial\theta^2} + \frac{\sqrt{2}}{2}\alpha|\beta|^2 + 2\bar{\partial}^*\bar{\partial}\alpha = 0.$$

Take scalar product with α and integrate along Σ by parts to get

$$-\int_\Sigma\langle\frac{\partial^2\alpha}{\partial\theta^2},\alpha\rangle + \frac{\sqrt{2}}{2}\int_\Sigma|\alpha|^2|\beta|^2 + 2\int_\Sigma|\bar{\partial}\alpha|^2 = 0,$$

for every $\theta\in[0,1]$. This equation makes sense in $\mathbb{S}^1$, since the values for $\theta = 0$ and $\theta = 1$ coincide. Then we can integrate again by parts to get

$$||\frac{\partial}{\partial\theta}\alpha||^2 + \frac{\sqrt{2}}{2}||\alpha\beta||^2 + 2||\bar{\partial}\alpha||^2 = 0.$$

So either $\alpha = 0$ or $\beta = 0$. In any case, A_θ, α_θ and β_θ are constant, i.e. if the line bundle L_Y admits any solutions to (2) then it is pulled-back from Σ and any solution is invariant under rotations in the $\mathbb{S}^1$ factor.

Assume now that $c_1(L_Y) = 2r\mathrm{P.D.}[\mathbb{S}^1]$. For any solution to (3), either $\alpha = 0$, $\bar{\partial}^*_{A_0}\beta = 0$ or $\beta = 0$, $\bar{\partial}_{A_0}\alpha = 0$. Also $2r = c_1(L_\Sigma) = \frac{\sqrt{-1}}{2\pi}\int_\Sigma F_A = \frac{1}{4\pi}\int_\Sigma(|\beta|^2 - |\alpha|^2)$. If $r < 0$ then $\beta = 0$ and the solutions to equations (3) are equivalent to the solutions to

$$\left\{ \begin{array}{l} \bar{\partial}_A\alpha = 0 \\ 2\sqrt{-1}\Lambda F_A = -|\alpha|^2 \end{array} \right.$$

on Σ. These are the typical vortex equations. The space of solutions is $s^d\Sigma$, where $d = g - 1 + r$. If $r < -(g-1)$ then there are no solutions. The case $r > 0$ is analogous. $\square$

Theorem 4.2. *Let $\mathfrak{s}_Y$ be a Spin$^{\mathbb{C}}$ structure on Y. Then $HFSW^*(Y,\mathfrak{s}_Y) = 0$ unless $c_1(\mathfrak{s}_Y) = 2r\mathrm{P.D.}[\mathbb{S}^1]$, with $-(g-1) \leq r \leq g-1$. Let $\mathfrak{s}_r$ be the Spin$^{\mathbb{C}}$ structure on Y with $c_1(\mathfrak{s}_r) = 2r\mathrm{P.D.}[\mathbb{S}^1]$, $-(g-1) \leq r \leq g-1$, $r \neq 0$. Put $d = g-1-|r| \geq 0$, then* $\dim HFSW^*(Y,\mathfrak{s}_r) \leq \dim H^*(s^d\Sigma)$.

Proof. The first claim is a direct consequence of proposition 4.1. Also from proposition 4.1, we know that the unperturbed Chern-Simons-Seiberg-Witten functional already has non-degenerate critical manifolds. As in [7] and [2, proposition 6], we can choose a perturbation modeled on the finite dimensional critical manifold $s^d\Sigma$. Choose a positive and perfect Morse function f on Σ,i.e. f has one critical point of index 0, $2g$ critical points of index 1 and one critical point of index 2. For any point $(x_1, x_2, \cdots, x_d) \in \Sigma$, define $F(x_1, x_2, \cdots, x_d) = \prod_{i=1}^d f(x_i)$, then it is easy to check that F is a Morse function on $s^d\Sigma$. The critical points of F consist of those $(x_1, x_2, \cdots, x_d)$ where x_i is a critical point of f, and the Morse index of $(x_1, x_2, \cdots, x_d)$ is the sum of the Morse indices of the x_i's. Therefore, the number of the critical points of F with Morse index i is given by

$$\binom{2g}{i} + \binom{2g}{i-2} + \cdots + \binom{2g}{i-2[i/2]},$$

which is exactly the i-th Betti number of $s^d\Sigma$ (see [8]). Hence, $F(x_1, x_2, \cdots, x_d)$ is a perfect Morse function on $s^d\Sigma$. Then we can perturb the Chern-Simons Seiberg-Witten functional such that there exists a one-to-one correspondence between the perturbed Seiberg-Witten monopoles on $\Sigma \times S^1$ and the critical points of F on $s^d\Sigma$. Both sets of critical points are non-degenerate and have the same relative indices modulo $2|r|$. This implies that $\dim HFSW^*(Y,\mathfrak{s}_r) \leq \dim H^*(s^d\Sigma)$. □

To shorten the notation, we shall write from now on

$$V_r = HFSW^*(Y,\mathfrak{s}_r), \tag{4}$$

for $-(g-1) \leq r \leq g-1$, $r \neq 0$. In this section we will study the finite dimensional vector spaces V_r for $r \neq 0$. They have a natural $\mathbb{Z}/2|r|\mathbb{Z}$-grading. The only tools we shall use are the bound on the dimension provided by theorem 4.2 and the gluing theorem 2.2. First, it is easily seen that the diffeomorphism $f \times c : \Sigma \times \mathbb{S}^1 \to \Sigma \times \mathbb{S}^1$, where $f : \Sigma \to \Sigma$ is an orientation reversing diffeomorphism and $c : \mathbb{S}^1 \to \mathbb{S}^1$ is the conjugation, induces an isomorphism $V_r \cong V_{-r}$. Henceforth we shall suppose $r > 0$ in (4).

Let $A = \Sigma \times D^2$ be the 4-manifold given as the product of Σ times a 2-dimensional disc, so that $\partial A = \Sigma \times \mathbb{S}^1$. Let $\Delta = \mathrm{pt} \times D^2 \subset A$. The Spin$^{\mathbb{C}}$ structures on A are parametrized by $H^2(A;\mathbb{Z}) = H^2(\Sigma;\mathbb{Z}) \cong \mathbb{Z}$. We write $\mathfrak{s}_r$ for the Spin$^{\mathbb{C}}$ structure on A with $c_1(\mathfrak{s}_r) = 2r\mathrm{P.D.}[\Delta]$ (we use the same name $\mathfrak{s}_r$ for Spin$^{\mathbb{C}}$ structures on Y and on A. No confussion should arise from this, as they are compatible in the sense that $\mathfrak{s}_r|_Y = \mathfrak{s}_r$). The relative Seiberg-Witten invariants of A give a map

$$\begin{array}{ccc} \mathbb{A}(\Sigma) & \to & V_r = HFSW^*(Y, \mathfrak{s}_r) \\ z & \mapsto & \phi_A^{SW}(\mathfrak{s}_r, z). \end{array} \tag{5}$$

As $S = A \cup_Y A = \Sigma \times \mathbb{S}^2$, the gluing theorem 2.2 yields

$$\sum_{n \in \mathbb{Z}} SW_{S, \mathfrak{s}_r + n[\Sigma]}(z_1 z_2) = \langle \phi_A^{SW}(\mathfrak{s}_r, z_1), \phi_A^{SW}(\mathfrak{s}_r, z_2) \rangle, \tag{6}$$

for any $z_1, z_2 \in \mathbb{A}(\Sigma)$, where the Spin$^{\mathbb{C}}$ structure $\mathfrak{s}_r$ on S is the one with $c_1(\mathfrak{s}_r) = 2r\text{P.D.}[\mathbb{S}^2]$. The metric that we must use for the Seiberg-Witten invariants in the left hand side of (6) is one giving a long neck, i.e. with period point ω_g close to $[\Sigma]$ in $\mathcal{K}_0 = \{a[\mathbb{S}^2] + b[\Sigma]/a, b > 0\}$. This implies that $c_1(\mathfrak{s}_r + n[\Sigma]) \cdot \omega_g > 0$ as $r > 0$, so the invariants are calculated in the component $-\mathcal{K}_0$ of the positive cone.

As $r \neq 0$, there is at most one $n \in \mathbb{Z}$ that contributes to the left hand side in (6), since $c_1(\mathfrak{s}_r + n[\Sigma]) = 2r[\mathbb{S}^2] + 2n[\Sigma]$ and

$$d(\mathfrak{s}_r + n[\Sigma]) = 2rn + 2(g-1).$$

It is thus important to know the Seiberg-Witten invariants of $S = \Sigma \times \mathbb{S}^2$ for the component $-\mathcal{K}_0$, which we describe now. We fix the homology orientation given by the usual orientation of $H^1(S) = H^1(\Sigma)$ and the orientation of $H^2_+(S) = \mathbb{R}\omega_g$ determined by $-\omega_g$.

Fix a symplectic basis $\{\gamma_1, \ldots, \gamma_{2g}\}$ of $H_1(\Sigma; \mathbb{Z})$ with $\gamma_i \gamma_{i+g} = \text{pt}$, for $1 \leq i \leq g$. Then $\mathbb{A}(\Sigma) = \mathbb{Q}[x] \otimes \Lambda^*(\gamma_1, \ldots, \gamma_{2g})$ and there is an action of the mapping class group of Σ, $\pi_0(\text{Diff}(\Sigma))$, factoring through an action of the symplectic group $\text{Sp}\,(2g, \mathbb{Z})$, on both $\mathbb{A}(\Sigma)$ and V_r, making the map (5) equivariant. Put $\theta = \sum_{i=1}^g \gamma_i \gamma_{g+i}$. Then the invariant part $\mathbb{A}(\Sigma)_I$ of $\mathbb{A}(\Sigma)$ is generated by x and θ. We decompose $\mathbb{A}(\Sigma)$ in irreducible $\text{Sp}\,(2g, \mathbb{Z})$-representations as

$$\mathbb{A}(\Sigma) = \bigoplus_{k=0}^{g} \Lambda_0^k \otimes \frac{\mathbb{Q}[x, \theta]}{(\theta^{g+1-k})},$$

where $\Lambda_0^k = \Lambda_0^k H_1(\Sigma) = \ker(\theta^{g-k+1} : \Lambda^k H_1(\Sigma) \to \Lambda^{2g-k+2} H_1(\Sigma))$ is the primitive component of $\Lambda^k H_1(\Sigma)$, for $0 \leq k \leq g$. Then, as the Seiberg-Witten invariant $SW_{S,\mathfrak{s}}(z)$ is invariant under the action of $\text{Diff}(\Sigma)$, $SW_{S,\mathfrak{s}}(z) = 0$ for any $z \in \bigoplus_{k=1}^{g} \Lambda_0^k \otimes \mathbb{Q}[x, \theta]/(\theta^{g+1-k})$, and it only matters to compute $SW_{S,\mathfrak{s}}(z)$ for $z = x^a \theta^b$.

Lemma 4.3. *Fix $0 < r \leq g-1$ and $n \in \mathbb{Z}$. Set $d = g-1-r$. Then $SW_{S,\mathfrak{s}_r + n[\Sigma]}$ is zero unless $n \leq -1$ and $D = rn + g - 1 \geq 0$ (there is only a finite number of such n). In that case $SW_{S,\mathfrak{s}_r + n[\Sigma]}(x^a \theta^b) = \frac{g!}{(g-b)!}(-n)^{g-b}$, for $a + b = D$, $0 \leq b \leq g$. Note that $D \leq d$ and $D \equiv d \pmod{r}$. As a consequence, for $n = -1$ (i.e. $D = d$) we have $SW_{S,\mathfrak{s}_r - [\Sigma]}(z) = \langle z, [s^d \Sigma] \rangle$, for any $z \in \mathbb{A}(\Sigma)$ of degree $2d$.*

Proof. Let L be the determinant bundle of $\mathfrak{s}_r+n[\Sigma]$, so that $c_1(L)=2r[\mathbb{S}^2]+2n[\Sigma]$. Let $H=\Sigma+\epsilon\mathbb{S}^2$ be a polarisation close to $[\Sigma]$, i.e. $\epsilon>0$ small. Then $\deg_H L=2r+2n\epsilon>0$, so by [3, proposition 27] the non-perturbed Seiberg-Witten moduli space on S is $\mathbb{P}(H^0(K\otimes\mathcal{L}^\vee)^*)$, where $-K+2\mathcal{L}=L$, so $K-\mathcal{L}=\frac{K-L}{2}\equiv(g-1-r)[\mathbb{S}^2]+(-1-n)[\Sigma]$. For $n\geq 0$ this is empty and hence $SW_{S,\mathfrak{s}_r+n[\Sigma]}=0$.

For $n\leq -1$, $d(\mathfrak{s}_r+n[\Sigma])=2(rn+g-1)$. Let $H_0=\epsilon\Sigma+\mathbb{S}^2$ be a polarisation close to $\mathbb{S}^2$, i.e. $\epsilon>0$ small. Then $\deg_{H_0}L=2r\epsilon+2n<0$, so by [3, proposition 27] the non-perturbed Seiberg-Witten moduli space on S is $\mathbb{P}(H^0(\mathcal{L})^*)$, where $\mathcal{L}=\frac{K+L}{2}\equiv(g-1+r)[\mathbb{S}^2]+(-1+n)[\Sigma]$. Hence the moduli space is empty and the Seiberg-Witten invariant for this polarisation, is zero. The Seiberg-Witten invariant $SW_{S,\mathfrak{s}_r+n[\Sigma]}$ is obtained via wall-crossing from [21]. With the notations therein, $u_c\in\Lambda^2H_1(S;\mathbb{Z})$ given by $u_c(\gamma_i\wedge\gamma_j)=\frac{1}{2}\langle\gamma_i\cup\gamma_j,c_1(L)\rangle=n$ and

$$SW_{S,\mathfrak{s}_r+n[\Sigma]}(x^a\theta^b)=\langle\theta^b\frac{(-u_c)^{g-b}}{(g-b)!},[\mathrm{Jac}S]\rangle=\frac{g!}{(g-b)!}(-n)^{g-b}.$$

The sign is as stated as there is a minus sign coming in as we compute the invariants in the component $-\mathcal{K}_0$ and another minus sign because we orient H^2_+ with $-\omega_g$.

The last statement follows from [8]. □

The last ingredient that we need in order to describe the vector space V_r is the cohomology $H^*(s^d\Sigma)$ of the d-th symmetric product $s^d\Sigma$ of the surface Σ. Here $d=g-1-r$, so d is in the range $0\leq d<g-1$. This cohomology ring was initially described in [8] and revisited in [20, section 4] where it was described as $\mathrm{Sp}\,(2g,\mathbb{Z})$-representation, which is the form well suited for our purposes. Put an auxiliary complex structure on Σ and interpret $s^d\Sigma$ as the moduli space of degree d effective divisors on Σ. Let $D\subset s^d\Sigma\times\Sigma$ be the universal divisor. Then $\psi_i=c_1(D)/\gamma_i\in H^1(s^d\Sigma)$, $1\leq i\leq 2g$, and $\eta=c_1(D)/x\in H^2(s^d\Sigma)$ are generators of the ring $H^*(s^d\Sigma)$, i.e. there is a graded $\mathrm{Sp}\,(2g,\mathbb{Z})$-equivariant epimorphism

$$\mathbb{A}(\Sigma)\cong\mathbb{Q}[\eta]\otimes\Lambda^*(\psi_1,\dots,\psi_{2g})\twoheadrightarrow H^*(s^d\Sigma). \tag{7}$$

Clearly η and $\theta=\sum_{i=1}^g\psi_i\psi_{g+i}$ generate the invariant part $H^*(s^d\Sigma)_I$. The description of $H^*(s^d\Sigma)$ as $\mathrm{Sp}\,(2g,\mathbb{Z})$-representation is given in the following

Proposition 4.4. *(*[20, proposition 4.2]*) For $0\leq d\leq g-1$ there is a presentation*

$$H^*(s^d\Sigma)=\bigoplus_{k=0}^d\Lambda_0^k\otimes\frac{\mathbb{Q}[\eta,\theta]}{J_k^g},$$

where $J_k^g=(R_k^g,\theta R_{k+1}^g,\theta^2R_{k+2}^g,\dots,\theta^{d+1-k})$, $0\leq k\leq d$, and

$$R_k^g=\sum_{i=0}^{\alpha}\frac{\binom{(d-k)-\alpha+1}{i}}{\binom{g-k}{i}}\frac{(-\theta)^i}{i!}\eta^{\alpha-i},$$

for $0 \leq k \leq d$, *with* $\alpha = [\frac{d-k}{2}] + 1$ *(for consistency,* $R^g_{d+1} = 1$*). Actually* $J^g_k = (R^g_k, \theta R^g_{k+1})$*. A basis for* $\mathbb{Q}[\eta, \theta]/J^g_k$ *as vector space is given by* $\eta^a \theta^b$*, with* $2a + b \leq d - k$*.* □

Corollary 4.5. *Fix* $0 < r \leq g - 1$ *and put* $d = g - 1 - r$*. Let* $z_i \in \mathbb{A}(\Sigma)$*,* $i \in I$*, be homogeneous elements such that* $\{z_i\}_{i \in I}$ *is a basis for* $H^*(s^d\Sigma)$*, under the epimorphism* (7)*. Consider for each* $i \in I$ *the element* $e_i = \phi^{SW}_A(\mathfrak{s}_r, z_i) \in V_r = HFSW^*(Y, \mathfrak{s}_r)$*. Then* $\{e_i\}_{i \in I}$ *is a basis for* V_r*. Therefore* $H^*(s^d\Sigma) \to V_r$*,* $z_i \mapsto e_i$*, is a (*$Sp(2g, \mathbb{Z})$*-equivariant) isomorphism of vector spaces.*

Proof. Without loss of generality, we may suppose that $\{z_i\}_{i \in I}$ is a basis formed by homogeneous elements with non-decreasing degrees. The intersection matrix $(\langle z_i, z_j \rangle)$ is then of the form

$$\begin{pmatrix} 0 & \cdots & 0 & A_0 \\ 0 & \cdots & A_1 & 0 \\ \vdots & \ddots & \vdots & \vdots \\ A_{2d} & \cdots & 0 & 0 \end{pmatrix}$$

where A_i are submatrices corresponding to the intersection product $H^i(s^d\Sigma) \otimes H^{2d-i}(s^d\Sigma) \to \mathbb{Q}$. So $\det A_i \neq 0$, for $0 \leq i \leq 2d$. By the formula (6) and lemma 4.3, $\langle e_i, e_j \rangle = 0$ if $\deg z_i + \deg z_j > 2d$ and $\langle e_i, e_j \rangle = \langle z_i, z_j \rangle$ if $\deg z_i + \deg z_j = 2d$. Therefore the intersection matrix $(\langle e_i, e_j \rangle)$ is of the form

$$\begin{pmatrix} * & \cdots & * & A_0 \\ * & \cdots & A_1 & 0 \\ \vdots & \ddots & \vdots & \vdots \\ A_{2d} & \cdots & 0 & 0 \end{pmatrix}$$

which is invertible. This implies in particular that $\dim V_r \geq \dim H^*(s^d\Sigma)$. As we already have the opposite inequality from theorem 4.2, it must be $\dim V_r = \dim H^*(s^d\Sigma)$ and $\{e_i\}_{i \in I}$ is a basis for V_r. □

Criterium 4.6. Let $z \in \mathbb{A}(\Sigma)$. Then the following are equivalent:

- $\phi^{SW}_A(\mathfrak{s}_r, z) = 0$.
- $SW_{S, \mathfrak{s}_r + n[\Sigma]}(z z_i) = 0$ for all $i \in I$ and integer n.
- $SW_{S, \mathfrak{s}_r + n[\Sigma]}(z z') = 0$ for all $z' \in \mathbb{A}(\Sigma)$ and integer n.

5. Ring structure of $HFSW^*(\Sigma \times \mathbb{S}^1, \mathfrak{s}_r)$ for $r \neq 0$

Recall our basic set up. We have the three manifold $Y = \Sigma \times \mathbb{S}^1$ together with the Spin$^{\mathbb{C}}$ structure $\mathfrak{s}_r$ with $c_1(\mathfrak{s}_r) = 2r\text{P.D.}[\mathbb{S}^1] \in H^2(Y; \mathbb{Z})$, $0 < r \leq g - 1$, and put $d = g - 1 - r$. In V_r we can define a ring structure by considering the cobordism X between $Y \sqcup Y$ and Y coming from taking the product of the pair of pants (a sphere with three holes) times the surface Σ. Let $\mathfrak{s}_r$ be the $\mathrm{Spin}^{\mathbb{C}}$ structure on X pulled back from that of Σ. Endowing X with three cylindrical ends $Y \times [0,\infty)$, we can define a map

$$\begin{array}{ccc} CFSW_*(Y) \otimes CFSW_*(Y) & \to & CFSW_*(Y) \\ a \otimes b & \mapsto & \sum_c \#\mathcal{M}^0(a,b;c)\, c \end{array} \tag{8}$$

where $\mathcal{M}^0(a,b;c)$ stands for the zero dimensional moduli space of (perturbed) Seiberg-Witten solutions on X with asymptotic limits a, b and c in the three ends, respectively. Note that generically $\mathcal{M}^0(a,b;c)$ consists of finitely many oriented regular points, so $\#\mathcal{M}^0(a,b;c)$ is well-defined. Then (8) descends to homology to give a map

$$\cdot : HFSW_*(Y,\mathfrak{s}_r) \otimes HFSW_*(Y,\mathfrak{s}_r) \to HFSW_*(Y,\mathfrak{s}_r).$$

This map satisfies $\phi^{SW}_A(\mathfrak{s}_r,z_1)\cdot\phi^{SW}_A(\mathfrak{s}_r,z_2) = \phi^{SW}_A(\mathfrak{s}_r,z_1z_2)$, for any $z_1,z_2 \in \mathbb{A}(\Sigma)$, thus defining an associative and graded commutative ring structure on V_r such that (5) is a ring epimorphism.

Remark 5.1. We can define the product on V_r only using criterium 4.6 as follows. By criterium 4.6, $\mathcal{I}_g = \{z \in \mathbb{A}(\Sigma) |\, \phi^{SW}_A(\mathfrak{s}_r,z) = 0\}$ is an ideal of $\mathbb{A}(\Sigma)$. So we define the ring structure of V_r by $\phi^{SW}_A(\mathfrak{s}_r,z_1)\cdot\phi^{SW}_A(\mathfrak{s}_r,z_2) = \phi^{SW}_A(\mathfrak{s}_r,z_1z_2)$, for any $z_1,z_2 \in \mathbb{A}(\Sigma)$. Therefore $V_r = \mathbb{A}(\Sigma)/\mathcal{I}_g$.

We set

$$\begin{cases} \eta = \phi^{SW}_A(\mathfrak{s}_r,x) \in V_r \\ \psi_i = \phi^{SW}_A(\mathfrak{s}_r,\gamma_i) \in V_r, \qquad 1 \le i \le 2g \end{cases}$$

where η has degree 2 and ψ_i degree 1. These are generators of V_r as algebra. This means that (5) is a $\mathrm{Sp}\,(2g,\mathbb{Z})$-equivariant epimorphism

$$\mathbb{A}(\Sigma) \cong \mathbb{Q}[\eta] \otimes \Lambda^*(\psi_1,\dots,\psi_{2g}) \twoheadrightarrow V_r.$$

Clearly η and $\theta = \sum_{i=1}^g \psi_i\psi_{g+i}$ generate the invariant part of V_r. Now we are going to relate the ring structure of $H^*(s^d\Sigma)$ with that of V_r. Note that the isomorphism $V_r \cong V_{-r}$ intertwines the ring structures, so we may restrict to the case $r > 0$. Recall that $d = g-1-r$.

Theorem 5.2. *Denote by $\cdot$ the product induced in $H^*(s^d\Sigma)$ by the product in V_r under the isomorphism of corollary 4.5. Then $\cdot$ is a deformation of the cup product graded modulo $2r = 2(g-1-d)$, i.e. for $f_1 \in H^i(s^d\Sigma)$, $f_2 \in H^j(s^d\Sigma)$, it is $f_1 \cdot f_2 = \sum_{m\ge 0}\Phi_m(f_1,f_2)$, where $\Phi_m \in H^{i+j+2mr}(s^d\Sigma)$ and $\Phi_0 = f_1 \cup f_2$.*

Proof. By lemma 4.3, for any $i,j \in I$, $\langle e_i,e_j\rangle$ is zero unless $\deg z_i + \deg z_j = 2d - 2mr$, with $m \ge 0$. Moreover, when $\deg z_i + \deg z_j = 2d$, it is $\langle e_i,e_j\rangle = \langle z_i,z_j\rangle$. Now the same argument as in [17, theorem 5] accomplishes the result, the only difference being that, in our present case, the deformation produces terms of increasing degrees. □

Corollary 5.3. *Let $f \in \mathbb{A}(\Sigma)$ be an homogeneous element of degree strictly bigger than $2d$. Then f is zero in V_r.* □

Proposition 5.4. *Let $0 < r \leq g-1$ and set $d = g-1-r$. Then there is a presentation*

$$V_r = \bigoplus_{k=0}^{d} \Lambda_0^k \otimes \frac{\mathbb{Q}[\eta,\theta]}{I_k^g},$$

where $I_k^g = (\mathcal{R}_k^g, \theta\mathcal{R}_{k+1}^g) \subset \mathbb{Q}[\eta,\theta]$ are ideals (dependent on g, k and r) such that

$$\mathcal{R}_k^g = R_k^g + \sum_{\substack{i=2\alpha+2mr-(d-k) \\ m>0}}^{\alpha+mr} \frac{a_{im}}{i!\binom{g-k}{i}} \eta^{\alpha+mr-i}\theta^i, \tag{9}$$

where $\alpha = [\frac{d-k}{2}]+1$, R_k^g are given in proposition 4.4 and a_{im} are some complex numbers (dependent on g, k, r). A basis for $\mathbb{Q}[\eta,\theta]/I_k^g$ is given by $\eta^a\theta^b$, with $2a+b \leq d-k$.

Proof. Let $\{z_i^{(k)}\}$ be a basis for Λ_0^k. Then by proposition 4.4, $z_i^{(k)}x^a\theta^b$, $2a+b+k \leq d$, form a basis for $H^*(s^d\Sigma)$. We use this basis in corollary 4.5 to construct a ($\mathrm{Sp}\,(2g,\mathbb{Z})$-equivariant) isomorphism $H^*(s^d\Sigma) \cong V_r$. The fact that $R_k^g \in J_k^g$ means that $z_i^{(k)}R_k^g = 0$ in $H^*(s^d\Sigma)$. Fix $z_0^{(k)} = \psi_1 \cdots \psi_k \in \Lambda_0^k$, then $\Lambda_0^k = \mathrm{Span} < \mathrm{Sp}\,(2g,\mathbb{Z}) z_0^{(k)} >$. Rewriting $z_0^{(k)}R_k^g$ in terms of the product $\cdot$ of theorem 5.2, and using the arguments of [25, section 2] (and the fact that the action of $\mathrm{Sp}\,(2g,\mathbb{Z})$ respects the ring structure on V_r), we get that

$$z_0^{(k)}R_k^g + \sum_{m>0} z_i^{(k)} R_{kim}^g = 0 \tag{10}$$

in V_r, where $\deg R_{kim}^g = \deg R_k^g + mr = \alpha + mr$, and R_{kim}^g is expressible in terms of the chosen basis, i.e. as a linear combination of the monomials $\eta^{\alpha+mr-j}\theta^j$, for $2\alpha+2mr-(d-k) \leq j \leq \alpha+mr$. As in the proof of [18, proposition 16], we have that the only nonvanishing R_{kim}^g in (10) correspond to $z_0^{(k)}$ (otherwise, one can find an element of $\mathrm{Sp}\,(2g,\mathbb{Z})$ only fixing $z_0^{(k)}$, this would produce a relation between the elements of the basis of V_r which is impossible), so (10) reduces to $z_0^{(k)}(R_k^g + \sum_{m>0} R_{k0m}^g) = 0$ in V_r. This produces the relation $\mathcal{R}_k^g = R_k^g + \sum_{m>0} R_{k0m}^g$ as stated in (9).

Also $\theta\mathcal{R}_{k+1}^g \in I_k^g$ since $\theta I_{k+1}^g \subset I_k^g$. Now I_k^g is generated by these two elements since J_k^g is generated by R_k^g and θR_{k+1}^g (see [25, section 2]). $\square$

Remark 5.5. Note that for d odd, $\mathcal{R}_k^g$ is the relation uniquely determined by expressing $\eta^\alpha \in \mathbb{Q}[\eta,\theta]/I_k^g$ in terms of the monomials of the basis. For d even, $\mathcal{R}_k^g$ is the relation uniquely determined by expressing $\eta^\alpha - \frac{(d-k)-\alpha+1}{g-k}\eta^{\alpha-1}\theta$ in terms of the monomials of the basis.

Corollary 5.6. *There is an isomorphism of the associated graded rings*

$$Gr_\theta V_r \cong Gr_\theta H^*(s^d\Sigma) \cong Gr_\gamma H_r,$$

where $d = g-1-|r|$ and $HF^*(\Sigma \times \mathbb{S}^1) = \oplus H_r$ is the instanton Floer homology of $Y = \Sigma \times \mathbb{S}^1$, H_r being the piece where α has eigenvalue $4r$ (if r is odd) or $4\sqrt{-1}r$ (if r is even), $-(g-1) \leq r \leq g-1, \neq 0$ (see [20, corollary 4.4]). $\square$

Lemma 5.7. *Fix $r>0$. Then the ideals of proposition 5.4 satisfy the recursion $I_k^g = I_{k-1}^{g-1}$, for $k>0$ and $r \le g-2$. Equivalently, $\mathcal{R}_k^g = \mathcal{R}_0^{g-k}$.*

Proof. By the computation of the Seiberg-Witten invariants of S in lemma 4.3 and the invariance under the action of $\mathrm{Sp}\,(2g,\mathbb{Z})$, we have

$$\begin{aligned} SW_{S,\mathfrak{s}_r+n[\Sigma]}(\gamma_1\cdots\gamma_k\gamma_{g+1}\cdots\gamma_{g+k}x^a\theta^b) &= SW_{S,\mathfrak{s}_r+n[\Sigma]}(\frac{1}{k!}\binom{g}{k}^{-1}\theta^k x^a\theta^b) \\ &= \frac{(g-k)!}{(g-k-b)!}(-n)^{g-k-b}, \end{aligned}$$

for $a+b=g-k-1-rn$. Therefore for any $R\in\mathbb{A}(\Sigma)_I$, $z\in\mathbb{A}(\Sigma)$,

$$\langle\psi_1\cdots\psi_{k-1}\psi_g R, \psi_{2g}z\rangle_g = \langle\psi_1\cdots\psi_{k-1}R, z\rangle_{g-1},$$

where the subindex means the genus of the surface Σ. This implies the statement. □

Now we aim to compute the coefficients a_{im} of $\mathcal{R}_0=\mathcal{R}_0^g$ in (9). Let $m>0$. We collect the coefficients together in a polynomial

$$p_m(x) = \sum a_{im}x^{g-i}, \tag{11}$$

where we consider $a_{im}=0$ for $i\notin[2\alpha+2mr-d, \alpha+mr]$, $\alpha=[\frac{d}{2}]+1$. Note that there are a finite number of non-zero polynomials. By analogy we consider

$$p_0(x) = (x-1)^{d-\alpha+1}x^{g-(d-\alpha-1)}, \tag{12}$$

so that $R_0=\sum\frac{a_{i0}}{i!\binom{g}{i}}\eta^{\alpha-i}\theta^i$, as given in proposition 4.4. By definition $\mathcal{R}_0=0\in V_r$, therefore we have $\langle\mathcal{R}_0,\eta^a\theta^b\rangle=0$, whenever $\alpha+a+b=d-kr$, $k\ge 0$. Now using the computation of the invariants of S in lemma 4.3, we get

$$\sum_{m=0}^k \frac{a_{im}}{i!\binom{g}{i}}\frac{g!}{(g-b-i)!}(k-m+1)^{g-b-i} = \sum_{m=0}^k a_{im}\frac{(g-i)!}{(g-b-i)!}(k-m+1)^{g-b-i}=0,$$

for all $k\ge 0$ and $0\le b\le d-\alpha-kr$. So

$$\sum_{m=0}^k \frac{d^b}{dx^b}p_m(x)\Big|_{x=k-m+1}=0.$$

for all $k\ge 0$ and $0\le b\le d-\alpha-kr$. By Taylor expansion, this is equivalent to saying that

$$p_k(x)\equiv -\left(p_0(x+k)+p_1(x+k-1)+\cdots+p_{k-1}(x+1)\right) \pmod{(x-1)^{d-\alpha-kr+1}}. \tag{13}$$

This condition, together with the fact that $p_k(x)$ has degree $g-(2\alpha+2kr-d)$ and it is divisible by $x^{g-(\alpha+kr)}$, uniquely determines $p_k(x)$ by recursion.

For instance, let us calculate explicitly $p_1(x)$. From (12) we have that

$$\begin{aligned} p_0(x+1) &= x^{d-\alpha+1}(x+1)^{g-(d-\alpha-1)} = x^{g-\alpha-r}(x+1)^{\alpha+r} \\ &= x^{g-\alpha-r}\sum_{k=0}^{\alpha+r}\binom{\alpha+r}{k}2^k(x-1)^{\alpha+r-k}, \end{aligned}$$

using that $d=g-1-r$. Now $p_1(x)$ is divisible by $x^{g-\alpha-r}$, has degree $(g-\alpha-r)+(d-\alpha-r)$ and $p_1(x)\equiv -p_0(x+1) \pmod{(x-1)^{d-\alpha-r+1}}$. Therefore

$$\begin{aligned} p_1(x) &= -x^{g-\alpha-r}\sum_{k=2\alpha+2r-d}^{\alpha+r}\binom{\alpha+r}{k}2^k(x-1)^{\alpha+r-k} = \\ &= -x^{g-\alpha-r}\sum_{\substack{2\alpha+2r-d\le k\le\alpha+r\\ 0\le j\le\alpha+r-k}} 2^k\binom{\alpha+r}{k}\binom{\alpha+r-k}{j}(-1)^j x^{\alpha+r-k-j} = \\ &= \sum_{\substack{2\alpha+2r-d\le k\le\alpha+r\\ 0\le j\le\alpha+r-k}} (-1)^{j+1}\frac{(\alpha+r)!}{k!j!(\alpha+r-k-j)!}2^k x^{g-k-j}. \end{aligned}$$

From this we may write the coefficients a_{i1} as

$$a_{i1} = \sum_{j=0}^{i-(2\alpha+2r-d)} (-1)^{j+1}\frac{(\alpha+r)!}{(i-j)!j!(\alpha+r-i)!}2^{i-j},$$

for $2\alpha+2r-d\le i\le\alpha+r$.

We can compute the rest of the coefficients a_{im}, for $m>1$, by recurrence using this method but the result is a collection of rather cumbersome formulae which do not shed light on the ring structure of V_r. This is to no surprise: the shape of the relations $\mathcal{R}^g_k$ depends on the basis of $\mathbb{Q}[\eta,\theta]/I^g_k$ that we have chosen in proposition 5.4, and this basis has been chosen rather arbritarily. We shall present now a slightly modified version of the previous argument which computes explicitly (a full set of) relations for V_r, by just not fixing any basis for $\mathbb{Q}[\eta,\theta]/I^g_k$. This leads to a closed formula for generators of the ideals I^g_k.

Theorem 5.8. *Let $0<r\le g-1$ and set $d=g-1-r$. Then there is a presentation*

$$V_r = \bigoplus_{k=0}^{d}\Lambda_0^k\otimes\frac{\mathbb{Q}[\eta,\theta]}{(\tilde{\mathcal{R}}^g_k,\theta\tilde{\mathcal{R}}^g_{k+1},\eta^{d+1},\theta^{d+1})},$$

where

$$\tilde{\mathcal{R}}^g_k = \sum_{i=0}^{\alpha}\frac{\binom{d-k-\alpha+1}{i}}{i!\binom{g-k}{i}}(-1)^i\eta^{\alpha-i}\theta^i - \sum_{i=0}^{\alpha+r}\frac{\binom{\alpha+r}{i}}{i!\binom{g-k}{i}}\eta^{\alpha+r-i}\theta^i,$$

where $\alpha=[\frac{d-k}{2}]+1$, for $0\le k\le d$, and $\tilde{\mathcal{R}}^g_{d+1}=1$.

Proof. By lemma 5.7 it is enough to find a relation for $k=0$,

$$\tilde{\mathcal{R}}_0 = R_0 + \sum_{\substack{m>0 \\ 0\le i\le \alpha+mr}} \frac{a_{im}}{i!\binom{g}{i}} \eta^{\alpha+mr-i}\theta^i, \tag{14}$$

This time we do not restrict the range for i. We only note that we can suppose $a_{im}=0$ if $i>g$, since $\theta^{g+1}=0$. As before, we collect the coefficients a_{im} of (14) in a polynomial $p_m(x)=\sum a_{im}x^{g-i}$, where $a_{im}=0$ for $i\notin[0,\alpha+mr]$. Also $p_0(x)=(x-1)^{d-\alpha+1}x^{g-(d-\alpha-1)}$. The condition that $\mathcal{R}_0$ be a relation is translated into

$$p_k(x) \equiv -\left(p_0(x+k)+p_1(x+k-1)+\cdots+p_{k-1}(x+1)\right) \pmod{(x-1)^{d-\alpha-kr+1}}. \tag{15}$$

We want to find polynomials $p_k(x)$ of degree g solving (15). This time the $p_k(x)$ are not determined uniquely, but we only need to find one solution. Since $p_0(x+1)=x^{d-\alpha+1}(x+1)^{g-(d-\alpha-1)}=x^{g-\alpha-r}(x+1)^{\alpha+r}$, we may choose

$$p_1(x) = -x^{g-\alpha-r}(x+1)^{\alpha+r}$$

and $p_k(x)=0$ for $k\ge 2$. This gives $a_{i1}=-\binom{\alpha+r}{i}$, $0\le i\le \alpha+r$, and $a_{im}=0$ for $m\ge 2$. In this way we have found $\tilde{\mathcal{R}}^g_k, \theta\tilde{\mathcal{R}}^g_{k+1}\in I^g_k$ as given in the statement. However they do not generate the whole ideal as may be seen by looking at the associated graded ring $\mathrm{Gr}_\theta\left(\mathbb{Q}[\eta,\theta]/(\tilde{\mathcal{R}}^g_k,\theta\tilde{\mathcal{R}}^g_{k+1})\right)$, so we need to add more relations. The nilpotence relations η^{d+1},θ^{d+1} are always satisfied by corollary 5.3. To see that these relations suffice, write any $f\in I^g_k$ as $f=a_1\mathcal{R}^g_k+a_2\theta\mathcal{R}^g_{k+1}$, by proposition 5.4. Then $f-a_1\tilde{\mathcal{R}}^g_k-a_2\theta\tilde{\mathcal{R}}^g_{k+1}\in I^g_k$ and has higher degree than that of f. Proceed recursively until we get a polynomial in $(\eta^{d+1},\theta^{d+1})$. □

Remark 5.9. Let us consider $\hat{V}_r = HFSW^*(Y,\mathfrak{s}_r)\otimes\mathbb{Q}((q))$, i.e. we introduce an extra variable q of degree $-|r|$ in the Seiberg-Witten-Floer homology, so that $\hat{V}_r$ is graded over $\mathbb{Z}$ (the grading is not canonically defined). In this way we homogenize the relations. Then

$$\hat{V}_r = \bigoplus_{k=0}^{d} \Lambda^k_0 \otimes \frac{\mathbb{Q}((q))[\eta,\theta]}{(\tilde{\mathcal{R}}^g_k,\theta\tilde{\mathcal{R}}^g_{k+1})}$$

where

$$\tilde{\mathcal{R}}^g_k = \sum_{i=0}^{\alpha} \frac{\binom{d-k-\alpha+1}{i}}{i!\binom{g-k}{i}}(-1)^i\eta^{\alpha-i}\theta^i - \sum_{i=0}^{\alpha+r}\frac{\binom{\alpha+r}{i}}{i!\binom{g-k}{i}}\eta^{\alpha+r-i}\theta^i q^2,$$

where $\alpha=[\frac{d-k}{2}]+1$, for $0\le k\le d$, and $\tilde{\mathcal{R}}^g_{d+1}=1$. The nilpotence relations are already included in this description. Indeed let $f\in\mathbb{A}(\Sigma)$ of degree $n>2d$. Then f goes to zero under (7), so that it can be written as qf_1 modulo the ideal $(\tilde{\mathcal{R}}^g_k,\theta\tilde{\mathcal{R}}^g_{k+1})$ where the degree of f_1 is $n+2|r|$. Proceeding recursively, we see that f lies in the intersection of the ideals $\bigcap_{n\ge 0}(q^n)=0$. So $f=0$.

6. Ring structure of $HFSW^*(\Sigma \times \mathbb{S}^1, \mathfrak{s}_0)$

In this section it is our intention to extend the arguments of sections 4 and 5 to compute $HFSW^*(Y, \mathfrak{s}_0)$, where $Y = \Sigma \times \mathbb{S}^1$ and $\mathfrak{s}_0$ stands for the $\mathrm{Spin}^{\mathbb{C}}$ structure with $c_1(\mathfrak{s}_0) = 0$.

As we know from section 3 we have to choose a perturbation of the form $t[\omega]$, $t > 0$ (equivalently, a perturbation of the form $t[-\omega]$, $t < 0$). We let $\omega \in H^2(Y;\mathbb{Z})$ be the pull-back of the standard symplectic form of Σ. The Poincaré dual is $C = -\mathbb{S}^1 = \mathrm{P.D.}[-\omega] \in H_1(Y;\mathbb{Z})$. We shall compute $HFSW^*_{t[\omega]}(Y, \mathfrak{s}_0)$, as a module over $\mathbb{Q}((q))$, where the subindex indicates the pertubation. The diffeomorphism $f \times c : \Sigma \times \mathbb{S}^1 \to \Sigma \times \mathbb{S}^1$, where $f : \Sigma \to \Sigma$ is an orientation reversing diffeomorphism and $c : \mathbb{S}^1 \to \mathbb{S}^1$ is the conjugation, induces an isomorphism $HFSW^*_{t[\omega]}(Y, \mathfrak{s}_0) \cong HFSW^*_{-t[\omega]}(Y, \mathfrak{s}_0)$ (changing also C to $-C$). As a shorthand, write $\hat{V}_0 = HFSW^*_{t[\omega]}(Y, \mathfrak{s}_0)$.

Proposition 6.1. *Let $\mathcal{M}$ be the moduli space of solutions to* (2) *with perturbation $\eta = t\omega$. Then $\mathcal{M}$ is Morse-Bott irreducible and isomorphic to $s^{g-1}\Sigma$. Then*

$$\dim_{\mathbb{Q}((q))} HFSW^*_{t[\omega]}(Y, \mathfrak{s}_0) \leq \dim H^*(s^{g-1}\Sigma).$$

Proof. The proof is similar to those of proposition 4.1 and theorem 4.2. The only differnece is that now the Seiberg-Witten-Floer chain complex takes its coefficients in $\mathbb{Q}((q))$. We omit the details. $\square$

Let $A = \Sigma \times D^2$ as before and choose $-\Delta = -\mathrm{pt} \times D^2 \subset A$ as a bounding manifold for $C = -\mathbb{S}^1 \subset Y$. In this situation the relative Seiberg-Witten invariants of A are defined in subsection 3.3 and give a map

$$\begin{array}{ccc} \mathbb{A}(\Sigma) & \to & \hat{V}_0 = HFSW^*_{t[\omega]}(Y, \mathfrak{s}_0) \\ z & \mapsto & \phi_A^{SW}(\mathfrak{s}_0, z). \end{array} \tag{16}$$

As $S = A \cup_Y A = \Sigma \times \mathbb{S}^2$, the gluing theorem 3.2 yields

$$\langle \phi_A^{SW}(\mathfrak{s}_0, z_1), \phi_A^{SW}(\mathfrak{s}_0, z_2) \rangle = \sum_{n \in \mathbb{Z}} SW_{S, \mathfrak{s}_0 + n[\Sigma]}(z_1 z_2) q^{-2n}, \tag{17}$$

for any $z_1, z_2 \in \mathbb{A}(\Sigma)$. The metric that we use for the Seiberg-Witten invariants of the right hand side of (17) has period point ω_g close to $[\Sigma]$ in $\mathcal{K}_0 = \{a[\mathbb{S}^2] + b[\Sigma] / a, b > 0\}$. The perturbation in S is $\omega = \mathrm{P.D.}[\mathbb{S}^2]$. Hence $(c_1(\mathfrak{s}_0 + n[\Sigma]) + t\omega) \cdot \omega_g > 0$ as $t > 0$, so the invariants are calculated in the component $-\mathcal{K}_0$ of the positive cone.

Lemma 6.2. *$SW_{S, \mathfrak{s}_0 + n[\Sigma]}$ is zero unless $n \leq -1$ (there is an infinite number of such n). In that case $SW_{S, \mathfrak{s}_0 + n[\Sigma]}(x^a \theta^b) = \frac{g!}{(g-b)!}(-n)^{g-b}$, for $a + b = D = g - 1$. For $n = -1$ we have $SW_{S, \mathfrak{s}_0 - [\Sigma]}(z) = \langle z, [s^{g-1}\Sigma] \rangle$, for any $z \in \mathbb{A}(\Sigma)$ of degree $2(g-1)$.*

Proof. This is worked out as in the proof of lemma 4.3. If L stands for the determinant bundle of $\mathfrak{s}_0+n[\Sigma]$ then for a polarisation H close to $[\Sigma]$, $(c_1(L)+t\omega)\cdot H = 2n\epsilon+t>0$. So for $n\geq 0$ we have $SW_{S,\mathfrak{s}_0+n[\Sigma]}=0$, by [3, proposition 27]. For $n\leq -1$, $d(\mathfrak{s}_0+n[\Sigma])=g-1$, and for a polarisation H_0 close to $\mathbb{S}^2$, $(c_1(L)+t\omega)\cdot H_0 = 2n+t\epsilon<0$ yielding the vanishing of the Seiberg-Witten invariant corresponding to this polarisation. Wall-crossing does the rest. □

Corollary 6.3. *Let $z_i\in\mathbb{A}(\Sigma)$, $i\in I$, be homogeneous elements such that $\{z_i\}_{i\in I}$ is a basis for $H^*(s^{g-1}\Sigma)$, under the epimorphism (7). Consider for each $i\in I$ the element $e_i=\phi^{SW}_A(\mathfrak{s}_0,z_i)\in\hat{V}_0$. Then $\{e_i\}_{i\in I}$ is a basis for $\hat{V}_0$ over $\mathbb{Q}((q))$. Therefore $H^*(s^{g-1}\Sigma)\otimes\mathbb{Q}((q))\to\hat{V}_0$, $z_i\mapsto e_i$, is a graded ($Sp(2g,\mathbb{Z})$-equivariant) isomorphism of vector spaces.* □

To define a ring structure on $\hat{V}_0=HFSW^*(Y,\mathfrak{s}_0)$, we consider the cobordism X between $Y\sqcup Y$ and Y which is the product of the pair of pants times the surface Σ. Let $\mathfrak{s}_0$ be the trivial Spin$^{\mathbb{C}}$ structure on X Endowing X with three cylindrical ends $Y\times[0,\infty)$, the moduli space of (perturbed) Seiberg-Witten solutions on X with asymptotic limits a, b and c in the three ends, respectively, is denoted by $\mathcal{M}(a,b;c)$. Generically, $\mathcal{M}(a,b;c)$ is a smooth manifold of fixed dimension with infinitely many components. There is a natural 2-cycle D in X which is a sphere with three holes bounded by $S^1\sqcup S^1$ and $-S^1$ in $Y\sqcup Y$ and Y. Define the energy map as the relative Chern class of $[A,\Phi]$ restricted to $(D,\partial D)$,

$$\mathcal{E}:\mathcal{M}(a,b;c)\to H^2(D,\partial D;\mathbb{Z})\cong\mathbb{Z},$$

$\mathcal{E}([A,\Phi])=\langle[\frac{\sqrt{-1}}{2\pi}F_A],[D]\rangle$, which is bounded from below. Denote by $\mathcal{M}^{(n)}(a,b;c)$ the preimage of $n\in\mathbb{Z}$ under $\mathcal{E}$. For $\mathrm{ind}(a)+\mathrm{ind}(b)=\mathrm{ind}(c)$, each $\mathcal{M}^{(n)}(a,b;c)$ consists of finitely many oriented points, hence yielding a well-defined map:

$$\begin{array}{ccc} CFSW^i_{t[\omega]}(Y,\mathfrak{s}_0)\otimes CFSW^j_{t[\omega]}(Y,\mathfrak{s}_0) & \to & CFSW^{i+j}_{t[\omega]}(Y,\mathfrak{s}_0) \\ a\otimes b & \mapsto & \sum\limits_c \#\mathcal{M}^{(n)}(a,b;c)q^n\, c \end{array} \tag{18}$$

It is standard to see that (18) descends to homology to give a map $\cdot:\hat{V}_0\otimes\hat{V}_0\to\hat{V}_0$. This map satisfies $\phi^{SW}_A(\mathfrak{s}_0,z_1)\cdot\phi^{SW}_A(\mathfrak{s}_0,z_2)=\phi^{SW}_A(\mathfrak{s}_0,z_1z_2)$, for any $z_1,z_2\in\mathbb{A}(\Sigma)$, thus defining an associative and graded commutative ring structure on $\hat{V}_0$ such that (16) is a ring epimorphism.

We define

$$\begin{cases} \eta=\phi^{SW}_A(\mathfrak{s}_0,x)\in\hat{V}_0 \\ \psi_i=\phi^{SW}_A(\mathfrak{s}_r,\gamma_i)\in\hat{V}_0, \qquad 1\leq i\leq 2g \end{cases}$$

where η has degree 2, ψ_i has degree 1 and q has degree 0. The generators of $\hat{V}_0$ as algebra over $\mathbb{Q}((q))$ are η and ψ_i, $1\leq i\leq 2g$. η and $\theta=\sum_{i=1}^g\psi_i\psi_{g+i}$ generate the invariant part.

Analogous to theorem 5.2, corollary 5.3, corollary 5.6, theorem 5.8, proposition 5.4 and remark 5.9, we have the following results on the ring structure of $\hat{V}_0$.

Proposition 6.4. *Denote by $\cdot$ the product induced in $H^*(s^{g-1}\Sigma)\otimes\mathbb{Q}((q))$ by the product in $\hat{V}_0$ under the isomorphism of Corollary 6.3. Then $\cdot$ is a deformation of the cup product, i.e. for $f_1 \in H^i(s^{g-1}\Sigma)$, $f_2 \in H^j(s^{g-1}\Sigma)$, it is $f_1 \cdot f_2 = \sum_{m\geq 0}\Phi_m(f_1,f_2)q^{2m}$, where $\Phi_m \in H^{i+j}(s^{g-1}\Sigma)$ and $\Phi_0 = f_1 \cup f_2$.* □

Corollary 6.5. *Let $f \in \mathbb{A}(\Sigma)$ be an homogeneous element of degree strictly bigger than $2(g-1)$. Then f is zero in $\hat{V}_0$.* □

Proposition 6.6. *There is a presentation*

$$\hat{V}_0 = \bigoplus_{k=0}^{g-1}\Lambda_0^k \otimes \frac{\mathbb{Q}((q))[\eta,\theta]}{(\mathcal{R}_k^g, \theta\mathcal{R}_{k+1}^g)},$$

where

$$\mathcal{R}_k^g = R_k^g + \sum_{\substack{i=2\alpha-(g-1-k)\\ m>0}}^{\alpha} \frac{a_{im}}{i!\binom{g-k}{i}}\eta^{\alpha-i}\theta^i q^{2m},$$

where $\alpha = [\frac{g-1-k}{2}]+1$, R_k^g are given in proposition 4.4 and a_{im} are some complex numbers (dependent on g and k). □

Theorem 6.7. *There is a presentation*

$$\hat{V}_0 = \bigoplus_{k=0}^{g-1}\Lambda_0^k \otimes \frac{\mathbb{Q}((q))[\eta,\theta]}{(\tilde{\mathcal{R}}_k^g, \theta\tilde{\mathcal{R}}_{k+1}^g)},$$

where

$$\tilde{\mathcal{R}}_k^g = \sum_{i=0}^{\alpha}\frac{\binom{g-1-k-\alpha+1}{i}}{i!\binom{g-k}{i}}(-1)^i\eta^{\alpha-i}\theta^i - \sum_{i=0}^{\alpha}\frac{\binom{\alpha}{i}}{i!\binom{g-k}{i}}\eta^{\alpha-i}\theta^i q^2,$$

where $\alpha = [\frac{g-1-k}{2}]+1$, for $0 \leq k \leq g-1$ and $\tilde{\mathcal{R}}_g^g = 1$. □

Corollary 6.8. *There is an isomorphism of the associated graded rings*

$$Gr_\theta\hat{V}_0 \cong Gr_\theta H^*(s^{g-1}\Sigma)\otimes\mathbb{Q}((q)) \cong Gr_\gamma H_0 \otimes \mathbb{Q}((q)),$$

where H_0 is the piece of the instanton Floer cohomology $HF^*(\Sigma\times\mathbb{S}^1)$ where α has eigenvalue 0 (see [20, corollary 4.4]). *In particular, we have the following indentification between the instanton Floer cohomology and the Seiberg-Witten-Floer cohomology:*

$$Gr_\gamma HF^*(\Sigma\times\mathbb{S}^1)\otimes\mathbb{Q}((q)) \cong \bigoplus_{-(g-1)\leq r\leq(g-1)} Gr_\theta\hat{V}_r.$$

□

7. Seiberg-Witten invariants of connected sums along surfaces

We want to show, as a first application, how the knowledge of the previous sections can be used to compute the Seiberg-Witten invariants of four-manifolds which appear as connected sums along surfaces of other four-manifolds. This was first dealt with in a particular case in [13] to get a proof of the symplectic Thom conjecture. In the context of Donaldson invariants it has been extensively treated in [16] [19].

The set up is as follows (see [16]). Let $\bar{X}_1$ and $\bar{X}_2$ be smooth oriented 4-manifolds and let Σ be a compact oriented surface of genus $g \geq 1$. Suppose that we have embeddings $\Sigma \hookrightarrow \bar{X}_i$ with image Σ_i representing a *non-torsion* element in homology whose self-intersection is zero. This implies that $b^+ > 0$. Now take small closed tubular neighbourhoods N_{Σ_i} of Σ_i which are isomorphic to $A = \Sigma \times D^2$. Let X_i be the closure of $\bar{X}_i - N_{\Sigma_i}$, $i = 1, 2$. Then X_i is a 4-manifold with boundary $\partial X_i = Y = \Sigma \times \mathbb{S}^1$ and $\bar{X}_i = X_i \cup_Y A$. Take an identification $\phi : \partial X_1 \to -\partial X_2$ (i.e. an orientation reversing bundle isomorphism). We define the connected sum of $\bar{X}_1$ and $\bar{X}_2$ along Σ as

$$X = X(\phi) = X_1 \cup_\phi X_2.$$

The resulting 4-manifold depends in general on the isotopy class of ϕ, but we shall drop ϕ from the notation when there is no danger of confusion, and write then $X = \bar{X}_1 \#_\Sigma \bar{X}_2$. Consider Spin$^{\mathbb{C}}$ structures $\mathfrak{s}_i$ on X_i such that $\mathfrak{s}_1|_Y \cong -\mathfrak{s}_2|_Y \cong \mathfrak{s}_Y$, so that they can be glued together to get a Spin$^{\mathbb{C}}$ structure $\mathfrak{s}_o$ on X. The Spin$^{\mathbb{C}}$ structures $\mathfrak{s}$ such that $\mathfrak{s}|_{X_i} = \mathfrak{s}_i$, $i = 1, 2$, are those of the form $\mathfrak{s}_o + h$, where h is an element in the image of $H_2(Y;\mathbb{Z}) \to H_2(X;\mathbb{Z}) \cong H^2(X;\mathbb{Z})$, where the last map is Poincaré duality. Let $\mathcal{R}im \subset H^2(X;\mathbb{Z})$ be the subspace generated by the rim tori, i.e. the image of $H_1(\Sigma;\mathbb{Z}) \otimes [\mathbb{S}^1] \subset H_2(Y;\mathbb{Z})$ in $H^2(X;\mathbb{Z})$. Then any Spin$^{\mathbb{C}}$ structure $\mathfrak{s}$ such that $\mathfrak{s}|_{X_i} = \mathfrak{s}_i$, $i = 1, 2$, is of the form $\mathfrak{s} = \mathfrak{s}_o + h + n\Sigma$, where $h \in \mathcal{R}im$, $n \in \mathbb{Z}$.

Now fix $D \in H_2(X;\mathbb{Z})$ with $D \cdot \Sigma = n < 0$ and minimun, which exists since Σ_i represent both non-torsion homology classes. We can represent D by an embedded smooth surface $D \hookrightarrow X$ intersecting Y in a collection of disjoint circles. Therefore $D = D_1 + D_2$, where $D_i \subset X_i$ and $\partial D_1 = -\partial D_2 = n\mathbb{S}^1$. Close D_i with a colection of n horizontal slices $\Delta = \mathrm{pt} \times D^2 \subset A = \Sigma \times D^2$ to get $\bar{D}_i = D_i + n\Delta \subset \bar{X}_i = X_i \cup_Y A$. Then an element $h \in \mathcal{R}im$ satisfies $h \cdot D = 0$, whereas $\Sigma \cdot D = n$.

If $\mathfrak{s}_Y$ has $c_1(\mathfrak{s}_Y)$ distinct to rP.D.$[\mathbb{S}^1]$, for any $-(g-1) \leq r \leq g-1$, then theorem 4.2 tells us that $SW_{X,\mathfrak{s}} = 0$. Now consider the case $\mathfrak{s}_Y = \mathfrak{s}_r$, with $-(g-1) \leq r \leq g-1$. Set $d = g - 1 - |r|$ as usual. We have two cases, $r \neq 0$ or $r = 0$.

Theorem 7.1. *Suppose $r \neq 0$. Fix $z_i \in \mathbb{A}(\Sigma)$, $i \in I$, homogeneous elements such that $\{z_i\}_{i \in I}$ is a basis for $H^*(s^d\Sigma)$. Then there exists a universal matrix $(m_{ij})_{i,j \in I}$ such that for every connected sum $X = \bar{X}_1 \#_\Sigma \bar{X}_2$ along a surface of genus g, Spin$^{\mathbb{C}}$ structures $\bar{\mathfrak{s}}_i$ on $\bar{X}_i$*

with $c_1(\bar{\mathfrak{s}}_i)\cdot\Sigma = 2r$ and Spin$^{\mathbb{C}}$ structure $\mathfrak{s}_o$ on X obtained by gluing $\bar{\mathfrak{s}}_1$ and $\bar{\mathfrak{s}}_2$, we have

$$\sum_{h\in\mathcal{R}im} SW_{X,\mathfrak{s}_o+h}(z_1z_2) = \sum_{\substack{n,m\in\mathbb{Z}\\ i,j\in I}} m_{ij} SW_{\bar{X}_1,\bar{\mathfrak{s}}_1+n\Sigma}(z_1z_i)\cdot SW_{\bar{X}_2,\bar{\mathfrak{s}}_2+m\Sigma}(z_2z_j),$$

for any $z_1\in\mathbb{A}(\bar{X}_1)$ and $z_2\in\mathbb{A}(\bar{X}_2)$ with $d(\mathfrak{s}_o)=\deg z_1+\deg z_2$ (note that at most one n and one m appear in every summand of the right hand side). If any of the manifolds involved has $b^+=1$ then its Seiberg-Witten invariants are computed for the component of the positive cone containing $-rP.D.[\Sigma]$.

Proof. Let $\mathfrak{s}_i = \bar{\mathfrak{s}}_i|_{X_i}$, $i=1,2$. By corollary 4.5, the elements $e_i=\phi_A^{SW}(\mathfrak{s}_r,z_i)$, $i\in I$, form a basis for $V_r = HFSW^*(Y,\mathfrak{s}_r)$. Therefore $V_r\to\mathbb{R}^{|I|}$, given as $\phi\mapsto(\langle\phi,\phi_A^{SW}(\mathfrak{s}_r,z_i)\rangle)_{i\in I}$, is an isomorphism such that

$$\phi_{X_1}^{SW}(\mathfrak{s}_1,z_1)\mapsto\left(\sum_{n\in\mathbb{Z}} SW_{\bar{X}_1,\bar{\mathfrak{s}}_1+n\Sigma}(z_1z_i)\right)_{i\in I}.$$

Theorem 2.2 says that $\sum_n SW_{\bar{X}_1,\bar{\mathfrak{s}}_1+n\Sigma}(z_1z_i) = \langle\phi_{X_1}^{SW}(\mathfrak{s}_1,z_1),e_i\rangle$, $\sum_m SW_{\bar{X}_2,\bar{\mathfrak{s}}_2+m\Sigma}(z_2z_j) = \langle\phi_{X_2}^{SW}(\mathfrak{s}_2,z_2),e_j\rangle$ and

$$\sum_{\{\mathfrak{s}/\mathfrak{s}|_{X_i}=\mathfrak{s}_i,\ i=1,2\}} SW_{X,\mathfrak{s}}(z_1z_2) = \langle\phi_{X_1}^{SW}(\mathfrak{s}_1,z_1),\phi_{X_2}^{SW}(\mathfrak{s}_2,z_2)\rangle.$$

Only the Spin$^{\mathbb{C}}$ structures of the form $\mathfrak{s}=\mathfrak{s}_o+h$, $h\in\mathcal{R}im$, satisty $d(\mathfrak{s})=\deg z_1+\deg z_2$. The result follows with (m_{ij}) being the inverse of the intersection matrix for the basis $\{e_i\}_{i\in I}$. Note that this matrix is explicitly computable, since by lemma 4.3 the products $\langle e_i,e_j\rangle$ are known. □

Theorem 7.2. *Suppose $r=0$. Fix $z_i\in\mathbb{A}(\Sigma)$, $i\in I$, homogeneous elements such that $\{z_i\}_{i\in I}$ is a basis for $H^*(s^{g-1}\Sigma)$. Then there exists universal matrices $(m_{r,ij})_{i,j\in I}$, for $r\geq r_0$ (for some integer r_0), such that for every connected sum $X=\bar{X}_1\#_\Sigma\bar{X}_2$ along a surface of genus g, Spin$^{\mathbb{C}}$ structures $\bar{\mathfrak{s}}_i$ on $\bar{X}_i$ with $c_1(\bar{\mathfrak{s}}_i)\cdot\Sigma=0$ and Spin$^{\mathbb{C}}$ structure $\mathfrak{s}_o$ on X obtained by gluing $\bar{\mathfrak{s}}_1$ and $\bar{\mathfrak{s}}_2$, such that $c_1(\mathfrak{s}_o)\cdot D = c_1(\bar{\mathfrak{s}}_1)\cdot\bar{D}_1 + c_1(\bar{\mathfrak{s}}_2)\cdot\bar{D}_2$ we have*

$$\sum_{h\in\mathcal{R}im} SW_{X,\mathfrak{s}_o+h}(z_1z_2) = \sum_{\substack{n,m\in\mathbb{Z}\\ i,j\in I}} m_{2(n+m)\Sigma\cdot D,ij} SW_{\bar{X}_1,\bar{\mathfrak{s}}_1+n\Sigma}(z_1z_i)\cdot SW_{\bar{X}_2,\bar{\mathfrak{s}}_2+m\Sigma}(z_2z_j),$$

for any $z_1\in\mathbb{A}(\bar{X}_1)$ and $z_2\in\mathbb{A}(\bar{X}_2)$ with $d(\mathfrak{s}_o)=\deg z_1+\deg z_2$ (only a finite number of n and m contribute to the right hand side). If any of the manifolds involved has $b^+=1$ then its Seiberg-Witten invariants are computed for the component of the positive cone containing $-P.D.[\Sigma]$.

Proof. We work as before using the gluing theorem 3.2,

$$\sum_{n\in\mathbb{Z}} SW_{\bar{X}_1,\bar{\mathfrak{s}}_1+n\Sigma}(z_1z_i)q^{c_1(\bar{\mathfrak{s}}_1+n\Sigma)\cdot\bar{D}_1} = \langle\phi_{X_1}^{SW}(\mathfrak{s}_1,z_1),e_i\rangle,$$

$$\sum_{m\in\mathbb{Z}} SW_{\bar X_2,\bar{\mathfrak{s}}_2+m\Sigma}(z_2z_j)q^{c_1(\bar{\mathfrak{s}}_2+m\Sigma)\cdot\bar D_2}=\langle\phi^{SW}_{X_2}(\mathfrak{s}_2,z_2),e_j\rangle,$$

$$\sum_{\{\mathfrak{s}/\mathfrak{s}|_{X_i}=\mathfrak{s}_i,\ i=1,2\}} SW_{X,\mathfrak{s}}(z_1z_2)q^{c_1(\mathfrak{s})\cdot D}=\langle\phi^{SW}_{X_1}(\mathfrak{s}_1,z_1),\phi^{SW}_{X_2}(\mathfrak{s}_2,z_2)\rangle,$$

to obtain

$$\sum_{\{\mathfrak{s}/\mathfrak{s}|_{X_i}=\mathfrak{s}_i,\ i=1,2\}} SW_{X,\mathfrak{s}}(z_1z_2)q^{c_1(\mathfrak{s})\cdot D}=$$

$$=\sum_{\substack{n,m\in\mathbb{Z},r\geq r_0\\ i,j\in I}} m_{r,ij}q^rSW_{\bar X_1,\bar{\mathfrak{s}}_1+n\Sigma}(z_1z_i)\cdot SW_{\bar X_2,\bar{\mathfrak{s}}_2+m\Sigma}(z_2z_j)q^{c_1(\bar{\mathfrak{s}}_1+n\Sigma)\cdot\bar D_1+c_1(\bar{\mathfrak{s}}_2+m\Sigma)\cdot\bar D_2}.$$

where $\{\mathfrak{s}/\mathfrak{s}|_{X_i}=\mathfrak{s}_i,\ i=1,2\}=\{\mathfrak{s}_o+h+l\Sigma | h\in\mathcal{R}im, l\in\mathbb{Z}\}$ and the matrix $\left(\sum_r m_{r,ij}q^r\right)$ is the inverse of the intersection matrix for the basis $\{e_i\}_{i\in I}$ in $\mathbb{Q}((q))$. Equating the coefficients of every monomial gives the result. $\square$

Corollary 7.3. *If either of $\bar X_i$ has simple type then $\sum_{h\in\mathcal{R}im}SW_{X,\mathfrak{s}_o+h}(xz)=0$, for any $z\in\mathbb{A}(X)$ and Spin$^{\mathbb{C}}$ structure $\mathfrak{s}_o$ on X. Analogously, if either of $\bar X_i$ has strong simple type then $\sum_{h\in\mathcal{R}im}SW_{X,\mathfrak{s}_o+h}(z)=0$, for any $z\in\mathbb{A}(X)$ with $\deg(z)>0$ and any Spin$^{\mathbb{C}}$ structure $\mathfrak{s}_o$ on X.* $\square$

In order to remove the summatory over the subspace $\mathcal{R}im$ in corollary 7.3 we need an extra condition.

Definition 7.4. A connected sum $X=\bar X_1\#_\Sigma\bar X_2$ is admissible if $\mathcal{R}im$ has no torsion and there exists a subspace $V\subset H^2(X;\mathbb{Z})$ such that $H^2(X;\mathbb{Z})=V\oplus\mathcal{R}im$ and $c_1(\mathfrak{s})\in V$ for every basic class $\mathfrak{s}$ of X.

Remark 7.5. Suppose that $\bar X_1$ and $\bar X_2$ are Kähler surfaces and $\Sigma_i\subset\bar X_i$ are smooth complex curves of genus g, isomorphic as complex curves, such that there is a deformation Kähler family $\mathcal{Z}\xrightarrow{\pi}D^2\subset\mathbb{C}$ with fiber $Z_t=\pi^{-1}(t)$, $t\neq0$, smooth and $Z_0=\pi^{-1}(0)=\bar X_1\cup_\Sigma\bar X_2$, the union of $\bar X_1$ and $\bar X_2$ along $\Sigma_1=\Sigma_2$ with a normal crossing. Then the general fiber $X=Z_t$ is the connected sum $X=\bar X_1\#_\Sigma\bar X_2$ with identification given by the isomorphism between the normal bundles of Σ_1 and Σ_2. If $H^2(X;\mathbb{Z})$ has no torsion then this identification is admissible, since for any basic class $\mathfrak{s}$ one has $c_1(\mathfrak{s})\in H^{1,1}$ and this space is orthogonal to $\mathcal{R}im$, as for any $T\in\mathcal{R}im$, it is $T^2=0$ and $\omega\cdot T=0$ (ω standing for the Kähler form). This implies that $T\notin H^{1,1}$ unless $T=0$.

Remark 7.6. In [12, definition 4.1], Morgan and Szabó define admissible identification when there exists a collection of primitive embedded (-2)-spheres in X (obtained by pasting embedded (-1)-discs in X_1 and X_2) generating a subspace $V\subset H_2(X)$ such that $H_2(X)=\mathcal{H}\oplus V$, where $\mathcal{H}=\{D\in H_2(X)/D|_Y=k[\mathbb{S}^1],\text{some } k\}$. Then $c_1(\mathfrak{s})\cdot V=0$ for any basic class $\mathfrak{s}$, and this implies admissibility in the sense of definition 7.4 (assuming again that $H^2(X;\mathbb{Z})$ has no torsion).

Corollary 7.7. *Suppose that the connected sum $X = \bar{X}_1 \# \bar{X}_2$ is admissible. If either of $\bar{X}_i$ has simple type then X has simple type. If either of $\bar{X}_i$ has strong simple type then X has strong simple type.* □

The formulas in theorems 7.1 and 7.2 become simpler when both $\bar{X}_i$ have $b_1 = 0$. We have the following result

Theorem 7.8. *Let $X = \bar{X}_1 \#_\Sigma \bar{X}_2$ be a connected sum along a surface of genus g where both $\bar{X}_1$ and $\bar{X}_2$ are of H_1-simple type. Let $\bar{\mathfrak{s}}_i$ be Spin$^{\mathbb{C}}$ structures on $\bar{X}_i$ with $c_1(\bar{\mathfrak{s}}_i)\cdot\Sigma = 2r$ and let $\mathfrak{s}_o$ be a Spin$^{\mathbb{C}}$ structure on X obtained by gluing $\bar{\mathfrak{s}}_1$ and $\bar{\mathfrak{s}}_2$, such that $c_1(\mathfrak{s}_o)\cdot D = c_1(\bar{\mathfrak{s}}_1)\cdot \bar{D}_1 + c_1(\bar{\mathfrak{s}}_2)\cdot\bar{D}_2$, with D, $\bar{D}_1$ and $\bar{D}_2$ as at the beginning of the section. Suppose $d = g-1-|r| \geq 0$. We have that $\sum_{h\in\mathcal{R}im} SW_{X,\mathfrak{s}_o+h}(z) =$*

$$\begin{cases} \sum\limits_{n,m\in\mathbb{Z}} (-1)^{d/2}\binom{g-1}{d/2} SW_{\bar{X}_1,\bar{\mathfrak{s}}_1+n\Sigma}(x^{d/2})\cdot SW_{\bar{X}_2,\bar{\mathfrak{s}}_2+m\Sigma}(x^{d/2}), & z=1,\ d \text{ even},\ r\neq 0 \\ \sum\limits_{n,m\in\mathbb{Z}} m_{2(n+m)\Sigma\cdot D} SW_{\bar{X}_1,\bar{\mathfrak{s}}_1+n\Sigma}(x^{d/2})\cdot SW_{\bar{X}_2,\bar{\mathfrak{s}}_2+m\Sigma}(x^{d/2}), & z=1,\ d \text{ even},\ r=0 \\ 0, & \text{otherwise.} \end{cases}$$

for some (universal) $m = \sum_{r\geq -1} m_{2r}q^{2r} \in \mathbb{Q}((q))$.

Proof. Let us suppose $r \neq 0$. By criterium 4.6, $\psi_j \phi^{SW}_{X_1}(\mathfrak{s}_1, z_1) = \phi^{SW}_{X_1}(\mathfrak{s}_1, \gamma_j z_1) = 0$, for $j = 1, \dots, 2g$. Therefore

$$\phi^{SW}_{X_1}(\mathfrak{s}_1, z_1) \in K = \bigcap_{1\leq j\leq 2g} \ker\psi_j,$$

for any $z_1 \in \mathbb{A}(X_1)$. Since $K \subset V_r$ is invariant under the action of $\mathrm{Sp}\,(2g,\mathbb{Z})$, we have that $K \subset \mathbb{Q}[\eta,\theta]/I^g_0$ by proposition 5.4. Now $f\in K$ if and only if $\psi_j f = 0$, for $j = 1,\dots,2g$. This means that $f \in I^g_1$ in the notation of proposition 5.4. So $K = I^g_1/I^g_0$, where the generators of I^g_1 are $\mathcal{R}^g_1$ and $\theta\mathcal{R}^g_2$. The intersection pairing $\langle,\rangle : K\otimes K \to \mathbb{Q}$ is the restriction of the pairing of V_r. Now by lemma 4.3 $\langle e_i, e_j\rangle = 0$ if $\deg z_i + \deg z_j > 2d$. For d odd, all the homogeneous components of all the elements in I^g_1 have degree strictly bigger than d (note that the component R^g_1 of $\mathcal{R}^g_1$ has degree $2([\frac{d-1}{2}]+1) = d+1$ and it is the component of lowest degree). So $K\otimes K\to\mathbb{Q}$ is the zero map for d odd, which proves the second line.

For d even, all the homogeneous components of all the elements in I^g_1 have degree strictly bigger than d, except for R^g_1, which has degree d. So $K\otimes K\to\mathbb{Q}$ has rank 1. Hence, for $z_1\in\mathbb{A}(\Sigma)$ and $z_2 \in \mathbb{A}(\Sigma)$, we have that

$$\sum_{h\in\mathcal{R}im} SW_{X,\mathfrak{s}_o+h}(z_1z_2) = \sum_{n,m\in\mathbb{Z}} c\, SW_{\bar{X}_1,\bar{\mathfrak{s}}_1+n\Sigma}(z_1x^{d/2}) SW_{\bar{X}_2,\bar{\mathfrak{s}}_2+m\Sigma}(z_2x^{d/2}),$$

where we have used $SW_{\bar{X}_i,\bar{\mathfrak{s}}_i+n\Sigma}(z_iR^g_1) = SW_{\bar{X}_i,\bar{\mathfrak{s}}_i+n\Sigma}(z_ix^{d/2})$, and where $c = \langle R^1_g, R^1_g\rangle^{-1}$. To compute c, note that $\theta R^1_g = 0$ so

$$\langle R^1_g, R^1_g\rangle = \langle R^1_g, \eta^{d/2}\rangle = \sum_{i=0}^{\alpha} \frac{\binom{d/2}{i}}{i!\binom{g-1}{i}}(-1)^i \frac{g!}{(g-i)!} = \sum_{i=0}^{\alpha}(-1)^i\binom{\alpha}{i}\frac{g}{g-i} = (-1)^\alpha \binom{g-1}{\alpha}^{-1},$$

with $\alpha = d/2$. Finally corollary 5.3 implies that $SW_{\bar{X}_i,\bar{\mathfrak{s}}_i+n\Sigma}(z_i x^{d/2}) = 0$ for any z_i with $\deg(z_i) > 0$. For $r = 0$, the proof is similar. $\square$

The following corollary is analogue to the result in [16, corollary 15] regarding the Kronheimer-Mrowka basic classes.

Corollary 7.9. *Suppose $\bar{X}_1$ is of strong simple type and $\bar{X}_2$ has H_1-simple type. Suppose that the connected sum $X = \bar{X}_1 \#_\Sigma \bar{X}_2$ is admissible. Then X is of strong simple type, there are no basic classes $\mathfrak{s}$ with $|c_1(\mathfrak{s}) \cdot \Sigma| < 2g-2$ and the basic classes for X are indexed by pairs of basic classes $(\bar{\mathfrak{s}}_1, \bar{\mathfrak{s}}_2)$ for $\bar{X}_1$ and $\bar{X}_2$ respectively, such that $\bar{\mathfrak{s}}_1 \cdot \Sigma = \bar{\mathfrak{s}}_2 \cdot \Sigma = \pm(2g-2)$.*
$\square$

8. Higher type adjunction inequalities

In this section we shall reprove the higher type adjunction inequalities for non-simple type 4-manifolds obtained by Oszváth and Szabó in [22]. Our method of proof is considerable simpler and parallels the proof of the higher type adjunction inequalities in the context of Donaldson invariants given in [20].

Proof of theorem 1.4. Without loss of generality, by reversing the orientation of Σ in the case $b^+ > 1$, we can suppose that $c_1(\mathfrak{s}) \cdot \Sigma \leq 0$. We reduce to the case of self-intersection zero by blowing-up. Let $N = \Sigma^2$ and consider the blow-up $\tilde{X} = X \# N\overline{\mathbb{CP}}^2$ with exceptional divisors $E_1, \ldots, E_N$. Let $\tilde{\Sigma} = \Sigma - E_1 - \cdots - E_N$ be the proper transform of Σ, which is an embedded surface of self-intersection zero and genus g, with $b \in \mathbb{A}(\tilde{\Sigma}) \cong \mathbb{A}(\Sigma)$. Consider the Spin$^{\mathbb{C}}$ structure $\tilde{\mathfrak{s}}$ on $\tilde{X}$ with $c_1(\tilde{\mathfrak{s}}) = c_1(\mathfrak{s}) - E_1 - \cdots - E_N$. Then $d(\mathfrak{s}) = d(\tilde{\mathfrak{s}})$,

$$-c_1(\tilde{\mathfrak{s}}) \cdot \tilde{\Sigma} + \tilde{\Sigma}^2 = -c_1(\mathfrak{s}) \cdot \Sigma + \Sigma^2,$$

and $SW_{\tilde{X},\tilde{\mathfrak{s}}}(a\, b) = SW_{X,\mathfrak{s}}(a\, b) \neq 0$.

Therefore we can suppose that $\Sigma^2 = 0$ and $c_1(\mathfrak{s}) \cdot \Sigma = -2r$, with $0 \leq r \leq g-1$. We prove the case $r > 0$ first. Let $\{\gamma_i\}$ be a symplectic basis of $H_1(\Sigma)$ with $\gamma_i \cdot \gamma_{g+i} = 1$, $1 \leq i \leq g$. Without loss of generality we may also suppose that $b = x^n \gamma_{i_1} \cdots \gamma_{i_m}$, $\deg(b) = 2n + m$. Now let $A = \Sigma \times D^2$ be a small tubular neighbourhood of $\Sigma \subset X$ and consider the splitting $X = X_1 \cup_Y A$, where X_1 is the closure of the complement of A and $\partial X_1 = \partial A = Y = \Sigma \times \mathbb{S}^1$. In this case $\mathfrak{s}$ is the only Spin$^{\mathbb{C}}$ structure appearing in the gluing formula in theorem 2.2, so

$$0 \neq SW_{X,\mathfrak{s}}(a\, b) = \langle \phi^{SW}_{X_1}(\mathfrak{s}, a), \phi^{SW}_A(\mathfrak{s}_{-r}, b) \rangle.$$

Then $\phi^{SW}_A(\mathfrak{s}_{-r}, b) \in HFSW^*(Y, \mathfrak{s}_{-r}) = V_{-r} \cong V_r$ is non-zero and therefore $\eta^n \psi_{i_1} \cdots \psi_{i_m} \neq 0 \in V_r$. By corollary 5.3, this implies $2n + m \leq 2d = 2(g-1-|r|)$. Therefore $2r + \deg(b) \leq 2g-2$.

For $r=0$, apply the gluing theorem 3.2

$$0 \neq \sum_{n\in\mathbb{Z}} SW_{X,\mathfrak{s}+n[\Sigma]}(a\,b)q^{c_1(\mathfrak{s}+n\Sigma)\cdot D} = \langle \phi^{SW}_{X_1}(\mathfrak{s},a), \phi^{SW}_A(\mathfrak{s}_0,b)\rangle.$$

Since the coefficient of $q^{c_1(\mathfrak{s})\cdot D}$, $SW_{X,\mathfrak{s}}(a\,b)$, is non-trivial, $\phi^{SW}_A(\mathfrak{s}_{-r},b)\in HFSW^*(Y,\mathfrak{s}_0)=\hat{V}_0$ is non-zero and therefore $\eta^n\psi_{i_1}\cdots\psi_{i_m}\neq 0\in\hat{V}_0$. By corollary 6.5, this implies $2n+m\leq 2d=2(g-1-|r|)$. Therefore $|c_1(\mathfrak{s})\cdot\Sigma|+\Sigma^2+\deg(b)\leq 2g-2$. □

Proof of theorem 1.6. Again we may suppose that Σ^2=0 and $c_1(\mathfrak{s})\cdot\Sigma=-2r$, with $0\leq r\leq g-1$. Let us do the case $r>0$. Suppose also that $b=x^n\gamma_{i_1}\cdots\gamma_{i_m}$, $\deg(b)=2n+m$. Now let $A=\Sigma\times D^2$ be a small tubular neighbourhood of $\Sigma\subset X$ and consider the splitting $X=X_1\cup_Y A$. Then

$$0\neq SW_{X,\mathfrak{s}}(a\,b)=\langle\phi^{SW}_{X_1}(\mathfrak{s},a),\phi^{SW}_A(\mathfrak{s}_{-r},b)\rangle.$$

Here $\phi^{SW}_{X_1}(\mathfrak{s},a)\in V_{-r}$ lives in the kernels of $\psi_1,\dots,\psi_l$, since as $\imath_*(\gamma_j)=0\in H_1(X)$,

$$\psi_j\phi^{SW}_{X_1}(\mathfrak{s},a)=\phi^{SW}_{X_1}(\mathfrak{s},\gamma_j a)=0,\qquad j=1,\dots,l.$$

Therefore it must be $\phi^{SW}_A(\mathfrak{s}_{-r},b)=\eta^n\psi_{i_1}\cdots\psi_{i_m}\notin(\psi_1,\dots,\psi_l)$ in V_{-r}. The argument in [20, proposition 3.7] (using proposition 5.4) shows that any element of degree bigger strictly bigger than $g-1-|r|$ must lie in the ideal $(\psi_1,\dots,\psi_{g-1-|r|})$ of V_{-r}. So if $l\geq g-1-|r|$ then $2n+m\leq g-1-|r|$, i.e. $\deg(b)\leq g-1-|r|$ and $|2r|+2\deg(b)\leq 2(g-1)$. If $l+1\leq g-1-|r|$ then obviously $|2r|+2\deg(b)\leq 2(g-1)$ as $\deg(b)\leq l+1$ by hypothesis. The case $r=0$ is similar using proposition 6.6. □

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